

# Asset Pricing

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## Modern Asset Pricing

- How do we value an arbitrary stream of future cash-flows?
- Equilibrium approach to the computation of asset prices. Rubinstein (1976) and Lucas (1978) tree model.
- Importance for macroeconomists: quantities *and* prices.
- We will work with a sequential markets structure with a complete set of Arrow securities.

## Household Utility

- Preferences:

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t))$$

- Budget constraints:

$$c_t(s^t) + \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) \leq e_t(s^t) + a_t(s^t)$$
$$-a_{t+1}(s^{t+1}) \leq A_{t+1}(s^{t+1})$$

## Problem of the Household

- We write the Lagrangian:

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left\{ \begin{array}{l} \beta^t \pi(s^t) u(c_t(s^t)) \\ + \lambda_t(s^t) \left( e_t(s^t) + a_t(s^t) - c_t(s^t) - \sum_{s_{t+1}} Q_t(s^t, s_{t+1}) a_{t+1}(s^t, s_{t+1}) \right) \\ + v_t(s^t) \left( A_{t+1}(s^{t+1}) + a_{t+1}(s^{t+1}) \right) \end{array} \right\}$$

- We take first order conditions with respect to  $c(s^t)$  and  $a_{t+1}(s^t, s_{t+1})$  for all  $s^t$ .
- Because of an Inada condition on  $u$ ,  $v_t(s^t) = 0$ .

## Solving the Problem

- FOCs for all  $s^t$ :

$$\begin{aligned}\beta^t \pi(s^t) u'(c_t(s^t)) - \lambda_t(s^t) &= 0 \\ -\lambda_t(s^t) Q_t(s^t, s_{t+1}) + \lambda_{t+1}(s_{t+1}, s^t) &= 0\end{aligned}$$

- Then:

$$Q_t(s^t, s_{t+1}) = \beta \pi(s_{t+1} | s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))}$$

- Fundamental Equation of Asset Pricing. Intuition.

## Interpretation

- The FOC is an equilibrium condition, not an explicit solution (we have endogenous variables in both sides of the equation).
- We need to evaluate consumption in equilibrium to obtain equilibrium prices.
- In our endowment set-up, this is simple.
- In production economies, it requires a bit more work.
- However, we already derived a moment condition that can be empirically implemented.

## The $j$ -Step Problem I

- How do we price claims further into the future?
- Create a new security  $a_{t+j}(s^t, s_{t+j})$ .
- For  $j > 1$ :

$$Q_t(s^t, s_{t+j}) = \beta^j \pi(s_{t+j} | s^t) \frac{u'(c_{t+j}(s^{t+j}))}{u'(c_t(s^t))}$$

- We express this price in terms of the prices of basic Arrow Securities.

## The $j$ -Step Problem II

- Manipulating expression:

$$\begin{aligned} Q_t(s^t, s_{t+j}) &= \beta^j \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \pi(s_{t+j}|s^{t+1}) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \frac{u'(c_{t+j}(s^{t+j}))}{u'(c_{t+1}(s^{t+1}))} \\ &= \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) Q_t(s^{t+1}, s_{t+j}) \end{aligned}$$

- Iterating:

$$Q_t(s^t, s_{t+j}) = \prod_{\tau=t}^{j-1} \sum_{s_{\tau+1}|s^\tau} Q_t(s^\tau, s_{\tau+1})$$

## The Stochastic Discount Factor

- Stochastic discount factor (SDF):

$$m_t(s^t, s_{t+1}) = \beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))}$$

- Note that:

$$\begin{aligned} \mathbb{E}_t m_t(s^t, s_{t+1}) &= \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) m_t(s^t, s_{t+1}) \\ &= \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \end{aligned}$$

- Interpretation of the SDF: discounting corrected by asset-specific risk.

## The Many Names of the Stochastic Discount Factor

The Stochastic discount factor is also known as:

1. Pricing kernel.
2. Marginal rate of substitution.
3. Change of measure.
4. State-dependent density.

## Pricing Redundant Securities I

- With our framework we can price any security (the  $j$ -step pricing was one of those cases).
- Contract that pays  $x_{t+1}(s^{t+1})$  in event  $s^{t+1}$ :

$$\begin{aligned} p_t(s_{t+1}, s^t) &= \beta \pi(s_{t+1} | s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} x_{t+1}(s^{t+1}) \\ &= \pi(s_{t+1} | s^t) m_t(s^t, s_{t+1}) x_{t+1}(s^{t+1}) \\ &= Q_t(s^t, s_{t+1}) x_{t+1}(s^{t+1}) \end{aligned}$$

## Pricing Redundant Securities II

- Contract that pays  $x_{t+1}(s^{t+1})$  in each event  $s^{t+1}$  (sum of different contracts that pay in one event):

$$\begin{aligned} p_t(s^t) &= \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} x_{t+1}(s^{t+1}) \\ &= \mathbb{E}_t m_t(s^t, s_{t+1}) x_{t+1}(s^{t+1}) \end{aligned}$$

- Note: we do not and we cannot take the expectation with respect to the price  $Q_t(s^t, s_{t+1})$ .

## Example I: Uncontingent One-Period Bond at Discount

- Many bonds are auctioned or sold at discount:

$$\begin{aligned} b_t(s^t) &= \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \\ &= \mathbb{E}_t m_t(s^t, s_{t+1}) \end{aligned}$$

- Then, the risk-free rate:

$$R_t^f(s^t) = \frac{1}{b_t(s^t)} = \frac{1}{\mathbb{E}_t m_t(s^t, s_{t+1})}$$

$$\text{or } \mathbb{E}_t m_t(s^t, s_{t+1}) R_t^f(s^t) = 1.$$

## Example II: One-Period Bond

- Other bonds are sold at face value:

$$1 = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} R_t^b(s^t) = \mathbb{E}_t m_t(s^t, s_{t+1}) R_t^b(s^t)$$

- As before, if the bond is risk-free:

$$1 = \mathbb{E}_t m_t(s^t, s_{t+1}) R_t^f(s^t)$$

### Example III: Zero-Cost Portfolio

- Short-sell an uncontingent bond and take a long position in a bond:

$$\begin{aligned} 0 &= \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} (R_t^b(s^t) - R_t^f(s^t)) \\ &= \mathbb{E}_t m_t(s^t, s_{t+1}) R_t^e(s^t) \end{aligned}$$

where  $R_t^e(s^t) = R_t^b(s^t) - R_t^f(s^t)$ .

- Why do we want to focus on excess returns? Different forces driven the risk-free interest rate and the risk premia.

## Example IV: Stock

- Buy at price  $p_t(s^t)$ , delivers a dividend  $d_{t+1}(s^{t+1})$ , sells at  $p_{t+1}(s^{t+1})$

$$p_t(s^t) = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} (p_{t+1}(s^{t+1}) + d_{t+1}(s^{t+1}))$$

- Often, we care about the price-dividend ratio (usually a stationary variable):

$$\frac{p_t(s^t)}{d_t(s^t)} = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \left( \frac{p_{t+1}(s^{t+1})}{d_{t+1}(s^{t+1})} + 1 \right) \frac{d_{t+1}(s^{t+1})}{d_t(s^t)}$$

## Example V: Options

- Call option: right to buy an asset at price  $K_1$ . Price of asset  $J(s^{t+1})$

$$co_t(s^t) = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \max \left( (J(s^{t+1}) - K_1) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))}, 0 \right)$$

- Put option: right to sell an asset at price  $K_1$ . Price of asset  $J(s^1)$

$$po_t(s^t) = \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \max \left( (K_1 - J(s^{t+1})) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))}, 0 \right)$$

## Example VI: Nominal Assets

- What happens if the price level,  $P(s^t)$  changes over time?
- We can focus on real returns:

$$\frac{p_t(s^t)}{P_t(s^t)} = \beta \sum_{s^{t+1}|s^t} \pi(s^{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \frac{x_{t+1}(s^{t+1})}{P_{t+1}(s^{t+1})} \Rightarrow$$

$$p_t(s^t) = \beta \sum_{s^1 \in S^1} \pi(s^1) \frac{u'(c(s^1))}{u'(c(s_0))} \frac{P_t(s^t)}{P_{t+1}(s^{t+1})} x_{t+1}(s^{t+1})$$

## Example VII: Term Structure of Interest Rates

- The risk-free rate  $j$  periods ahead is:

$$R_{tj}^f(s^t) = \left[ \beta^j \mathbb{E}_t \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \right]^{-1}$$

- And the yield to maturity is:

$$R_{tj}^{fy}(s^t) = \left( R_{tj}^f(s^t) \right)^{\frac{1}{j}} = \beta^{-1} \left[ u'(c_t(s^t)) \left( \mathbb{E}_t u'(c_{t+1}(s^{t+1})) \right)^{-1} \right]^{\frac{1}{j}}$$

- Structure of the yield curve.

## Non Arbitrage

- A lot of financial contracts are equivalent.
- From previous results, we derive a powerful idea: absence of arbitrage.
- Empirical evidence regarding non arbitrage.
- Possible limitations to non arbitrage conditions.
- Related idea: spanning of non-traded assets.

## A Numerical Example

- $u(c) = \log c, \beta = 0.99$
- $e(s^0) = 1, e(s_1 = high) = 1.1, e(s_1 = low) = 0.9.$
- $\pi(s_1 = high) = 0.5, \pi(s_2 = low) = 0.5.$

- Equilibrium prices:

$$q(s^0, s_1 = high) = 0.99 * 0.5 * \frac{1}{\frac{1}{1.1}} = 0.45$$

$$q(s^0, s_1 = low) = 0.99 * 0.5 * \frac{1}{\frac{1}{0.9}} = 0.55$$

$$q(s^0) = 0.45 + 0.55 = 1$$

- Note how the price is different from a naive adjustment by expectation and discounting:

$$q_{naive}(s^0, s_1 = high) = 0.99 * 0.5 * 1 = 0.495$$

$$q_{naive}(s^0, s_1 = low) = 0.99 * 0.5 * 1 = 0.495$$

$$q_{naive}(s^0) = 0.495 + 0.495 = 0.99$$

- Why is  $q(s^0, s_1 = high) < q(s^0, s_1 = low)$ ?
- Two forces:
  1. Discounting  $\beta$ .
  2. Ratio of marginal utilities:  $\frac{u'(c(s^1))}{u'(c(s_0))}$ .
- Covariance is key.

## Risk Correction I

We recall three facts:

$$1. Q_t(s^t, s_{t+1}) = \mathbb{E}_t m_t(s^t, s_{t+1}) = 1/R_t^f(s^t).$$

$$2. p_t(s^t) = \mathbb{E}_t m_t(s^t, s_{t+1}) x_{t+1}(s^{t+1}).$$

$$3. cov_t(x, y) = \mathbb{E}_t(xy) - \mathbb{E}_t(x)\mathbb{E}_t(y).$$

## Risk Correction II

Then:

$$p_t(s^t) = \mathbb{E}_t m_t(s^t, s_{t+1}) \mathbb{E}_t x_{t+1}(s^{t+1}) + \text{cov}_t(m_t(s^t, s_{t+1}), x_{t+1}(s^{t+1}))$$

or

$$\begin{aligned} p_t(s^t) &= \frac{\mathbb{E}_t m_t(s^t, s_{t+1})}{R_t^f(s^t)} + \text{cov}_t(m_t(s^t, s_{t+1}), x_{t+1}(s^{t+1})) \\ &= \frac{\mathbb{E}_t m_t(s^t, s_{t+1})}{R_t^f(s^t)} + \text{cov}_t\left(\beta \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))}, x_{t+1}(s^{t+1})\right) \\ &= \frac{\mathbb{E}_t m_t(s^t, s_{t+1})}{R_t^f(s^t)} + \beta \frac{\text{cov}(u'(c_{t+1}(s^{t+1})), x_{t+1}(s^{t+1}))}{u'(c_t(s^t))} \end{aligned}$$

## Covariance and Risk Correction I

Now we can see how if:

1. If  $\text{cov}_t \left( m_t \left( s^t, s_{t+1} \right), x_{t+1} \left( s^{t+1} \right) \right) = 0 \Rightarrow p_t \left( s^t \right) = \frac{\mathbb{E}_t m_t \left( s^t, s_{t+1} \right)}{R_t^f \left( s^t \right)}$ ,  
not adjustment for risk.
2. If  $\text{cov}_t \left( m_t \left( s^t, s_{t+1} \right), x_{t+1} \left( s^{t+1} \right) \right) > 0 \Rightarrow p_t \left( s^t \right) > \frac{\mathbb{E}_t m_t \left( s^t, s_{t+1} \right)}{R_t^f \left( s^t \right)}$ ,  
premium for risk (insurance).
3. If  $\text{cov}_t \left( m_t \left( s^t, s_{t+1} \right), x_{t+1} \left( s^{t+1} \right) \right) < 0 \Rightarrow p_t \left( s^t \right) < \frac{\mathbb{E}_t m_t \left( s^t, s_{t+1} \right)}{R_t^f \left( s^t \right)}$ ,  
discount for risk (speculation).

## Covariance and Risk Correction II

- Risk adjustment is  $cov_t \left( m_t \left( s^t, s_{t+1} \right), x_{t+1} \left( s^{t+1} \right) \right)$ .
- Basic insight: risk premium is generated by covariances, not by variances.
- Why? Because of risk aversion. Investor cares about volatility of consumption, not about the volatility of asset.
- For an  $\varepsilon$  change in portfolio:

$$\sigma^2 (c + \varepsilon x) = \sigma^2 (c) + 2\varepsilon cov (c, x) + \varepsilon^2 \sigma^2 (x)$$

## Utility Function and the Risk Premium

- We see how risk depends of marginal utilities:
  1. Risk-neutrality: if utility function is linear, you do not care about  $\sigma^2(c)$ .
  2. Risk-loving: if utility function is convex you want to increase  $\sigma^2(c)$ .
  3. Risk-averse: if utility function is concave you want to reduce  $\sigma^2(c)$ .
- It is plausible that household are (basically) risk-averse.

## A Small Detour

- Note that all we have said can be applied to the trivial case without uncertainty.
- In that situation, there is only one security, a bond, with price:

$$Q = \beta \frac{u'(c_{t+1})}{u'(c_t)}$$

- And the interest rate is:

$$R = \frac{1}{Q} = \frac{1}{\beta} \frac{u'(c_t)}{u'(c_{t+1})}$$

## Pricing Securities in the Solow Model

- Assume CRRA utility, that we are in a BGP with growth rate  $g$ , and define  $\beta = e^{-\delta}$ .
- Then:  $R = \frac{1}{\beta} \left( \frac{c}{(1+g)c} \right)^{-\gamma} = e^{\delta} (1+g)^{\gamma}$
- Or in logs:  $r \simeq \delta + \gamma g$ , where i.e., the real interest rate depends on the rate of growth of technology, the readiness of households to substitute intertemporally, and on the discount factor.
- Then,  $\gamma$  must be low to reconcile small international differences in the interest rate and big differences in  $g$ .

## More on the Risk Free Rate I

- Assume that the growth rate of consumption is log-normally distributed.
- Note that with a CRRA utility function:

$$R_t^f(s^t) = \frac{1}{\mathbb{E}_t m_t(s^t, s_{t+1})} = \frac{1}{\beta \mathbb{E}_t \left( \frac{c(s^{t+1})}{c(s^t)} \right)^{-\gamma}} = \frac{1}{\beta \mathbb{E}_t \left( e^{-\gamma \Delta \log c(s^{t+1})} \right)}$$

- Since  $\mathbb{E}_t(e^z) = e^{\mathbb{E}_t(z) + \frac{1}{2}\sigma^2(z)}$  if  $z$  is normal:

$$R_t^f(s^t) = \left[ \beta e^{-\gamma \mathbb{E}_t \Delta \log c(s^{t+1}) + \frac{1}{2} \gamma^2 \sigma^2(\Delta \log c(s^{t+1}))} \right]^{-1}$$

## More on the Risk Free Rate II

- Taking logs:

$$r_t^f(s^t) = \delta + \gamma \mathbb{E}_t \Delta \log c(s^{t+1}) - \frac{1}{2} \gamma^2 \sigma^2 (\Delta \log c(s^{t+1}))$$

- We can read this equation from right to left and from left to right!
- Rough computation (U.S. annual data, 1947-2005):
  1.  $\mathbb{E}_t \Delta \log c(s^{t+1}) = 0.0209$ .
  2.  $\sigma(\Delta \log c(s^{t+1})) = 0.011$ .
  3. Number for  $\gamma$ ? benchmark log utility  $\gamma = 1$ .

## Precautionary Savings

- Term  $\frac{\gamma^2}{2}\sigma^2 \left( \Delta \log c \left( s^{t+1} \right) \right)$  represents precautionary saving.

- Then, precautionary saving:

$$\frac{1^2}{2} (0.011)^2 = 0.00006 = 0.006\%$$

decreases the interest rate by a very small amount.

- Why decrease? General equilibrium effect.
- We will revisit this result when we talk about incomplete markets.
- Also,  $\frac{\gamma^2}{2}\sigma^2 \left( \Delta \log c \left( s^{t+1} \right) \right)$  is close to  $\frac{\gamma}{2}\sigma^2 \left( \log c \left( s^{t+1} \right) \right)$  (welfare cost of the business cycle):

$$\sigma^2 \left( \Delta \log c \left( s^{t+1} \right) \right) \approx 0.33 * \sigma^2 \left( \log c_{dev} \left( s^{t+1} \right) \right)$$

- We will come back to this in a few slides.

## Quadratic Utility

- Note that precautionary term appears because we use a CRRA utility function.
- Suppose instead that we have a quadratic utility function (Hall, 1978)

$$-\frac{1}{2}(a - c)^2$$

- Then:

$$R_t^f(s^t) = \frac{1}{\mathbb{E}_t m_t(s^t, s_{t+1})} = \frac{1}{\beta \mathbb{E}_t \left( \frac{a - c(s^{t+1})}{a - c(s^t)} \right)}$$

## Random Walk of Consumption I

- For  $a$  sufficiently big in relation with  $c(s^{t+1})$ :

$$\frac{a - c(s^{t+1})}{a - c(s^t)} \simeq 1 - \frac{1}{a} \Delta c(s^{t+1})$$

- Then:

$$R_t^f(s^t) = \frac{1}{e^{-\delta} \left(1 - \frac{1}{a} \mathbb{E}_t \Delta c(s^{t+1})\right)}$$

- Taking logs:  $r_t^f(s^t) = \delta + \frac{1}{a} \mathbb{E}_t \Delta c(s^{t+1})$ .

## Random Walk of Consumption II

- We derived Hall's celebrated result:

$$\mathbb{E}_t \Delta c (s^{t+1}) = a (r_t^f (s^t) - \delta)$$

- Consumption is a random walk (possibly with a drift).
- For the general case, we have a random walk in marginal utilities:

$$u' (c_t (s^t)) = \beta R_t^f (s^t) \mathbb{E}_t u' (c_{t+1} (s^{t+1}))$$

- Empirical implementation: GMM with additional regressors.
- Idea of Granger causality.

## Precautionary Behavior

- Difference between risk-aversion and precautionary behavior. Leland (1968), Kimball (1990).
- Risk-aversion depends on the second derivative (concave utility).
- Precautionary behavior depends on the third derivative (convex marginal utility).
- Relation with linearization and certainty equivalence.

## Random Walks I

- Can we predict the market?
- Remember that the price of a share was:

$$p_t(s^t) = \beta \sum_{s_{t+1}|s^t} \pi(s_{t+1}|s^t) \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} (p_{t+1}(s^{t+1}) + d_{t+1}(s^{t+1}))$$

or:

$$p_t(s^t) = \beta \mathbb{E}_t \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} (p_{t+1}(s^{t+1}) + d_{t+1}(s^{t+1}))$$

## Random Walks II

- Now, suppose that we are thinking about a short period of time, i.e.  $\beta \approx 1$  and that firms do not distribute dividends (historically not a bad approximation because of tax reasons):

$$p_t(s^t) = \beta \mathbb{E}_t \frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} (p_{t+1}(s^{t+1}))$$

- If in addition  $\frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))}$  does not change (either because utility is linear or because of low volatility of consumption):

$$p_t(s^t) = \mathbb{E}_t p_{t+1}(s^{t+1}) = p_t(s^t) + \varepsilon_{t+1}$$

## Random Walks III

- $p_t(s^t) = \mathbb{E}_t p_{t+1}(s^{t+1}) = p_t(s^t) + \varepsilon_{t+1}$  is a Random Walk.
- The best forecast of the price of a share tomorrow is today's price.
- Can we forecast future movements of the market? No!
- We can generalize the idea to other assets.
- Empirical evidence.

## A Second Look at Risk Correction I

- We can restate the previous result about risk correction in terms of returns.
- The pricing condition for a contract  $i$  with price 1 and yield  $R_t^i(s^{t+1})$  is:

$$1 = \mathbb{E}_t m_t(s^t, s_{t+1}) R_t^i(s^{t+1})$$

- Then:

$$1 = \mathbb{E}_t m_t(s^t, s_{t+1}) \mathbb{E}_t R_t^i(s^{t+1}) + cov_t(m_t(s^t, s_{t+1}), R_t^i(s^{t+1}))$$

## A Second Look at Risk Correction II

Multiplying by  $-R_t^f (s^t) = -\left(\mathbb{E}_t m_t (s^t, s_{t+1})\right)^{-1}$ :

$$\begin{aligned}
 \mathbb{E}_t R_t^i (s^{t+1}) - R_t^f (s^t) &= -R_t^f (s^t) \operatorname{cov}_t \left( m_t (s^t, s_{t+1}), R_t^i (s^{t+1}) \right) \\
 &= -R_t^f (s^t) \beta \frac{\operatorname{cov} \left( u' \left( c_{t+1} (s^{t+1}) \right), x_{t+1} (s^{t+1}) \right)}{u' (c_t (s^t))} \\
 &= -\frac{\operatorname{cov} \left( u' \left( c_{t+1} (s^{t+1}) \right), x_{t+1} (s^{t+1}) \right)}{\mathbb{E}_t u' \left( c_{t+1} (s^{t+1}) \right)}
 \end{aligned}$$

## Beta-Pricing Model

- Note

$$\begin{aligned}\mathbb{E}_t R_t^i (s^{t+1}) - R_t^f (s^t) &= -R_t^f (s^t) \text{cov}_t (m_t (s^t, s_{t+1}), R_t^i (s^{t+1})) \Rightarrow \\ \mathbb{E}_t R_t^i (s^{t+1}) &= R_t^f (s^t) + \left( \frac{\text{cov}_t (m_t (s^t, s_{t+1}), R_t^i (s^{t+1}))}{\sigma_t^2 (m_t (s^t, s_{t+1}))} \right) \left( -\frac{\sigma_t^2 (m_t (s^t, s_{t+1}))}{\mathbb{E}_t (m_t (s^t, s_{t+1}))} \right) \\ &= R_t^f (s^t) + \beta_{i,m,t} \lambda_{m,t}\end{aligned}$$

- Interpretation:

1.  $\beta_{i,m,t}$  is the quantity of risk of each asset (risk-free asset is the “zero-beta” asset).
2.  $\lambda_{m,t}$  is the market price of risk (same for all assets).

## Mean-Variance Frontier I

- Yet another way to look at the FOC:

$$1 = \mathbb{E}_t m_t (s^t, s_{t+1}) \mathbb{E}_t R_t^i (s^{t+1}) + cov_t (m_t (s^t, s_{t+1}), R_t^i (s^{t+1}))$$

- Then:

$$1 = \mathbb{E}_t m_t (s^t, s_{t+1}) \mathbb{E}_t R_t^i (s^{t+1}) + \frac{cov_t (m_t (s^t, s_{t+1}), R_t^i (s^{t+1}))}{\sigma_t (m_t (s^t, s_{t+1})) \sigma_t (R_t^i (s^{t+1}))} \sigma_t (m_t (s^t, s_{t+1})) \sigma_t (R_t^i (s^{t+1}))$$

## Mean-Variance Frontier II

- The coefficient of correlation between two random variables is:

$$\rho_{m,R_i,t} = \frac{\text{cov}_t \left( m_t \left( s^t, s_{t+1} \right), R_t^i \left( s^{t+1} \right) \right)}{\sigma_t \left( m_t \left( s^t, s_{t+1} \right) \right) \sigma_t \left( R_t^i \left( s^{t+1} \right) \right)}$$

- Then, we have:

$$1 = \mathbb{E}_t m_t \left( s^t, s_{t+1} \right) \mathbb{E}_t R_t^i \left( s^{t+1} \right) + \rho_{m,R_i,t} \sigma_t \left( m_t \left( s^t, s_{t+1} \right) \right) \sigma_t \left( R_t^i \left( s^{t+1} \right) \right)$$

- Or:

$$\mathbb{E}_t R_t^i \left( s^{t+1} \right) = R_t^f \left( s^t \right) - \rho_{m,R_i,t} \frac{\sigma_t \left( m_t \left( s^t, s_{t+1} \right) \right)}{\mathbb{E}_t m_t \left( s^t, s_{t+1} \right)} \sigma_t \left( R_t^i \left( s^{t+1} \right) \right)$$

## Mean-Variance Frontier III

- Since  $\rho_{m,R_i,t} \in [-1, 1]$  :

$$\left| \mathbb{E}_t R_t^i (s^{t+1}) - R_t^f (s^t) \right| \leq \frac{\sigma_t (m_t (s^t, s_{t+1}))}{\mathbb{E}_t m_t (s^t, s_{t+1})} \sigma_t (R_t^i (s^{t+1}))$$

- This relation is known as the *Mean-Variance frontier*: “How much return can you get for a given level of variance?”
- Any investor would hold assets within the mean-variance region.
- No assets outside the region will be hold.

## Market Price of Risk I

- $\frac{\sigma_t(m_t(s^t, s_{t+1}))}{\mathbb{E}_t m_t(s^t, s_{t+1})}$  can be interpreted as the market price of risk.
- Can we find a good approximation for the market price of risk?
- Assume a CRRA utility function. Then,  $m_t(s^t, s_{t+1}) = \beta \left( \frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)^{-\gamma}$

## A Few Mathematical Results

- Note that if  $z$  is normal

$$\begin{aligned}\mathbb{E}(e^z) &= e^{\mathbb{E}(z) + \frac{1}{2}\sigma^2(z)} \\ \sigma^2(e^z) &= \left(e^{\sigma^2(z)} - 1\right) e^{2\mathbb{E}(z) + \sigma^2(z)}\end{aligned}$$

hence

$$\frac{\sigma(e^z)}{\mathbb{E}(e^z)} = \left(\frac{\sigma^2(e^z)}{\mathbb{E}(e^z)^2}\right)^{0.5} = \left(e^{\sigma^2(z)} - 1\right)^{0.5}$$

- Also  $e^x - 1 \simeq x$

## Market Price of Risk II

- If we set  $z = \frac{1}{\beta} \log m_t (s^t, s_{t+1}) = -\gamma \log \left( \frac{c_{t+1}(s^{t+1})}{c_t(s^t)} \right)$ , we have:

$$\begin{aligned} \frac{\sigma_t \left( m_t (s^t, s_{t+1}) \right)}{\mathbb{E}_t m_t (s^t, s_{t+1})} &= \left( e^{\gamma^2 \sigma^2 (\Delta \ln c(s^{t+1}))} - 1 \right)^{0.5} \\ &\simeq \gamma \sigma \left( \Delta \ln c (s^{t+1}) \right) \end{aligned}$$

- Price of risk depends on EIS and variance of consumption growth.
- This term already appeared in our formula for the risk-free rate:

$$r_t^f (s^t) = \delta + \gamma \mathbb{E}_t \Delta \log c (s^{t+1}) - \frac{1}{2} \gamma^2 \sigma^2 \left( \Delta \log c (s^{t+1}) \right)$$

- Also, a nearly identical term,  $\frac{1}{2} \gamma \sigma^2 \left( \ln c_{dev} (s^{t+1}) \right)$ , was our estimate of the welfare cost of the business cycle.

## Link with Welfare Cost of Business Cycle I

- This link is not casual: welfare costs of uncertainty and risk price are two sides of the same coin.
- We can coax the cost of the business cycle from market data.
- In lecture 1, we saw that we could compute the cost of the business cycle by solving:

$$\mathbb{E}_{t-1} u \left[ (1 + \Omega_{t-1}) c(s^t) \right] = u \left( \mathbb{E}_{t-1} c(s^t) \right)$$

- Parametrize  $\Omega_{t-1}$  as a function of  $\alpha \in (0, 1)$ . Then:

$$\mathbb{E}_{t-1} u \left[ (1 + \Omega_{t-1}(\alpha)) c(s^t) \right] = \mathbb{E}_{t-1} u \left( \alpha \mathbb{E}_{t-1} c(s^t) + (1 - \alpha) c(s^t) \right)$$

## Link with Welfare Cost of Business Cycle II

- Take derivatives with respect to  $\alpha$  and evaluate at  $\alpha = 0$

$$\Omega'_{t-1}(0) = \frac{\mathbb{E}_{t-1} u' (c(s^t)) (\mathbb{E}_{t-1} c(s^t) - c(s^t))}{\mathbb{E}_{t-1} c(s^t) u' (c(s^t))}$$

- Dividing by  $\beta/u' (c(s^{t-1}))$ , we get  $m(s^t)$

$$\Omega'_{t-1}(0) = \frac{\mathbb{E}_{t-1} m_t (s^{t-1}, s_t) (\mathbb{E}_{t-1} c(s^t) - c(s^t))}{\mathbb{E}_{t-1} m_t (s^{t-1}, s_t) c(s^t)}$$

- Rearranging and using the fact that  $\Omega_{t-1}(0) = 0$ ,

$$1 + \Omega'_{t-1}(0) = \frac{\mathbb{E}_{t-1} m_t (s^{t-1}, s_t) \mathbb{E}_{t-1} c(s^t)}{\mathbb{E}_{t-1} m_t (s^{t-1}, s_t) c(s^t)}$$

## The Sharpe Ratio I

- Another way to represent the Mean-Variance frontier is:

$$\left| \frac{\mathbb{E}_t R_t^i (s^{t+1}) - R_t^f (s^t)}{\sigma_t (R_t^i (s^{t+1}))} \right| \leq \frac{\sigma_t (m_t (s^t, s_{t+1}))}{\mathbb{E}_t m_t (s^t, s_{t+1})}$$

- This relation is known as the *Sharpe Ratio*.
- It answers the question: “How much more mean return can I get by shouldering a bit more volatility in my portfolio?”
- Note again the market price of risk bounding the excess return over volatility.

## The Sharpe Ratio II

- For a portfolio at the Mean-Variance frontier:

$$\left| \frac{\mathbb{E}_t R_t^m (s^{t+1}) - R_t^f (s^t)}{\sigma_t (R_t^m (s^{t+1}))} \right| = \frac{\sigma_t (m_t (s^t, s_{t+1}))}{\mathbb{E}_t m_t (s^t, s_{t+1})}$$

- Given a CRRA utility function, we derive before that, for excess returns at the frontier:

$$\left| \frac{\mathbb{E}_t R_t^{me} (s^{t+1})}{\sigma_t (R_t^{me} (s^{t+1}))} \right| \simeq \gamma \sigma (\Delta \ln c (s^{t+1}))$$

- Alternatively (assuming  $\mathbb{E}_t R_t^m (s^{t+1}) > R_t^f (s^t)$ ):

$$\mathbb{E}_t R_t^{me} (s^{t+1}) \simeq R_t^f (s^t) + \gamma \sigma (\Delta \ln c (s^{t+1})) \sigma_t (R_t^m (s^{t+1}))$$

## The Equity Premium Puzzle II

- Let us go to the data and think about the stock market (i.e.  $R_t^i(s^{t+1})$  is the yield of an index) versus the risk free asset (the U.S. treasury bill).
- Average return from equities in XXth century: 6.7%. From bills 0.9%. (data from Dimson, Marsh, and Staunton, 2002).
- Standard deviation of equities: 20.2%.
- Standard deviation of  $\Delta \ln c(s^{t+1})$ : 1.1%.

## The Equity Premium Puzzle III

- Then:

$$\left| \frac{6.7\% - 0.9\%}{20.2\%} \right| = 0.29 \leq 0.011\gamma$$

that implies a  $\gamma$  of at least 26!

- But we argued before that  $\gamma$  is at most 10.
- This observation is known as the Equity Premium Puzzle (Mehra and Prescott, 1985).

## The Equity Premium Puzzle IV

- We can also look at the equity premium directly.
- Remember the beta formula:

$$\mathbb{E}_t R_t^{me} (s^{t+1}) \simeq R_t^f (s^t) + \gamma \sigma (\Delta \ln c (s^{t+1})) \sigma_t (R_t^m (s^{t+1}))$$

- Then

$$\gamma \sigma (\Delta \ln c (s^{t+1})) \sigma_t (R_t^m (s^{t+1})) = 0.011 * 0.202 * \gamma = 0.0022 * \gamma$$

- For  $\gamma = 3$ , the equity premium should be 0.0066.

## The Equity Premium Puzzle V

- Things are actually worse than they look:
  1. Correlation between individual and aggregate consumption is not one.
  2. However, U.S. treasury bills are also risky (inflation risk).
- We can redo the derivation of the Sharpe Ratio in terms of excess returns:

$$\left| \frac{\mathbb{E}_t R_t^e (s^{t+1})}{\sigma_t (R_t^e (s^{t+1}))} \right| \leq \frac{\sigma_t (m_t (s^t, s_{t+1}))}{\mathbb{E}_t m_t (s^t, s_{t+1})}$$

## The Equity Premium Puzzle VI

- Build a excess return portfolio (Campbell, 2003):

1. Mean: 8.1%

2. Standard deviation: 15.3%

- Then

$$\left| \frac{8.1\%}{15.3\%} \right| = 0.53 \leq 0.011\gamma$$

that implies a  $\gamma$  of at least 50!

## Raising $\gamma$

- A naive answer will be to address the equity premium puzzle by raising  $\gamma$  (Kandel and Stambaugh).
- We cannot really go ahead and set  $\gamma = 50$ :
  1. Implausible intercountry differences in real interest rates.
  2. We would generate a risk-free rate puzzle (Weil, 1989).

## The Risk-Free Rate Puzzle I

- Remember:

$$r_t^f(s^t) = \delta + \gamma \mathbb{E}_t \Delta \log c(s^{t+1}) - \frac{1}{2} \gamma^2 \sigma^2(\Delta \log c(s^{t+1}))$$

- $\Delta \log c(s^{t+1}) = 0.0209$ ,  $\sigma^2(\Delta \log c(s^{t+1})) = (0.011)^2$  and  $\gamma = 10$ :

$$\gamma \mathbb{E}_t \Delta \log c(s^{t+1}) - \frac{1}{2} \gamma^2 \sigma^2(\Delta \log c(s^{t+1})) = 10 * 2.09 - 0.5 * 100 * (0.011)^2 = 20.4\%$$

- Hence, even with  $r_t^f(s^t) = 4\%$ , we will need a  $\delta = -16.4\%$ : a  $\beta \gg 1$ !

## The Risk-Free Rate Puzzle II

- In fact, the risk-free rate puzzle is a problem by itself. Remember that rate of return on bills is 0.9%.

- $\Delta \log c(s^{t+1}) = 0.0209$ ,  $\sigma^2(\Delta \log c(s^{t+1})) = (0.011)^2$  and  $\gamma = 1$ :

$$0.009 = \delta + 0.0209 - \frac{1}{2}(0.011)^2$$

- This implies

$$\delta = 0.009 - 0.0209 + \frac{1}{2}(0.011)^2 = -0.0118$$

again, a  $\beta > 1$ !

## Answers to Equity Premium Puzzle

1. Returns from the market have been odd. If return from bills had been around 4% and returns from equity 5%, you would only need a  $\gamma$  of 6.25. Some evidence related with the impact of inflation (this also helps with the risk-free rate puzzle).
2. There were important distortions on the market. For example regulations and taxes.
3. Habit persistence.
4. Separating EIS from risk-aversion: Epstein-Zin preferences.
5. The model is deeply wrong: behavioral.

## Habit Persistence

- Assume that the utility function takes the form:

$$\frac{(c_t - hc_{t-1})^{1-\gamma} - 1}{1-\gamma}$$

- Interpretation. If  $h = 0$  we have our CRRA function back.
- External versus internal habit persistence.

## Why Does Habit Help? I

- Suppose  $c_{t+1}(s^{t+1}) = 1.01$ ,  $c_t(s^t) = c_{t-1}(s^{t-1}) = 1$ , and  $\gamma = 2$ :

$$\frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} = \frac{(1.01 - h)^{-2}}{(1 - h)^{-2}}$$

- If  $h = 0$

$$\frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} = \frac{(1.01)^{-2}}{(1)^{-2}} = 0.9803$$

- If  $h = 0.95$

$$\frac{u'(c_{t+1}(s^{t+1}))}{u'(c_t(s^t))} = \frac{(1.01 - 0.95)^{-2}}{(0.05)^{-2}} = 0.6944$$

## Why Does Habit Help? II

- In addition, there is an indirect effect, since we can raise  $\gamma$  without generating a risk-free rate puzzle.
- We will have:

$$\begin{aligned} R_t^f(s^t) &= \frac{1}{\mathbb{E}_t m_t(s^t, s_{t+1})} = \frac{1}{\beta \mathbb{E}_t \left( \frac{c(s^{t+1}) - hc(s^t)}{c(s^t) - hc(s^{t-1})} \right)^{-\gamma}} \\ &= \frac{1}{\beta \mathbb{E}_t \left( e^{-\gamma \Delta \log(c(s^{t+1}) - hc(s^t))} \right)} \end{aligned}$$

## Why Does Habit Help? II

- Now:

$$r_t^f(s^t) = \delta + \gamma \mathbb{E}_t \Delta \log(c(s^{t+1}) - hc(s^t)) - \frac{1}{2} \gamma^2 \sigma^2 (\Delta \log(c(s^{t+1}) - hc(s^t)))$$

- Note that for  $h$  close to 1

$$\mathbb{E}_t \Delta \log(c(s^{t+1}) - hc(s^t)) \approx \mathbb{E}_t \Delta \log(c(s^{t+1}))$$

- So we basically get a higher variance term, with a negative sign.
- Hence, we can increase the  $\gamma$  that will let us have a reasonable risk-free interest rate.

## Lessons from the Equity Premium Puzzle

We want to build DSGE models where the market price of risk is:

1. High.
2. Time-varying.
3. Correlated with the state of the economy.

We need to somehow fit together a low risk-free interest rate and a high return on risky assets.

## Main Ideas of Asset Pricing

1. Non-arbitrage.
2. Risk-free rate is  $r \simeq \delta + \gamma g$  + precautionary behavior.
3. Risk is not important by itself: the key is covariance.
4. Mean-Variance frontier.
5. Equity Premium Puzzle.
6. Random walk of asset prices.