Equilibrium with Complete Markets

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Arrow-Debreu versus Sequential Markets

- Previous lecture we discussed the preferences of agents in a situation with uncertainty.

- Now, we will discuss how market operates in a simple endowment economy.

- Two approaches: Arrow-Debreu and sequential markets.

- Under some technical conditions both approaches are equivalent.

Environment

- We will have I agents, $i = 1, ..., I$
- Endowment:

$$(e^1, ..., e^I) = \{e_t^1(s^t), ..., e_t^I(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

- Tradition in macro of looking at endowment economies. Why? Risk-sharing, consumption, asset pricing.
- Advantages and shortcomings.

Allocation

- An allocation is a sequence of consumption in each period and event for each individual:

$$(c^1, ..., c^I) = \{c_t^1(s^t), ..., c_t^I(s^t)\}_{t=0, s^t \in S^t}^{\infty}$$

- Feasible allocation: an allocation such that:

$$\begin{aligned} c_t^i(s^t) &\geq 0 \text{ for all } t, \text{ all } s^t \in S^t, \text{ for } i = 1, 2 \\ \sum_{i=1}^{I} c_t^i(s^t)(s^t) &\leq \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, \text{ all } s^t \in S^t \end{aligned}$$

Pareto Efficiency

- An allocation $\{(c_t^1(s^t), ..., c_t^I(s^t))\}_{t=0,s^t\in S^t}^{\infty}$ is Pareto efficient if it is feasible and if there is no other feasible allocation

$$\{(\tilde{c}_t^1(s^t), ..., \tilde{c}_t^I(s^t))\}_{t=0,s^t\in S^t}^{\infty}$$

such that

$$\begin{aligned} u(\tilde{c}^i) &\geq u(c^i) \text{ for all } i \\ u(\tilde{c}^i) &> u(c^i) \text{ for at least one } i \end{aligned}$$

- Ex ante efficiency.

Arrow-Debreu Market Structure

- Trade takes place at period 0, *before* any uncertainty has been realized (in particular, before s_0 has been realized).

- As for allocation and endowment, Arrow-Debreu prices have to be indexed by event histories in addition to time.

- Let $p_t(s^t)$ denote the price of one unit of consumption, quoted at period 0, delivered at period t if (and only if) event history s^t has been realized.

- We need to normalize one price to 1 and use it as numeraire.

Arrow-Debreu Equilibrium

A Arrow-Debreu equilibrium are prices $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ and allocations $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ such that:

1. Given $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$, for $i = 1, .., I$, $\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ solves

$$\max_{\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} \sum_{s^t\in S^t} p_t(s^t) c_t^i(s^t) \leq \sum_{t=0}^{\infty} \sum_{s^t\in S^t} p_t(s^t) e_t^i(s^t)$$

$$c_t^i(s^t) \geq 0 \text{ for all } t$$

2. Markets clear:

$$\sum_{i=1}^{I} \hat{c}_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, \text{ all } s^t \in S^t$$

Welfare Theorems

- Let $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ be a competitive equilibrium allocation. Then $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ is Pareto efficient.

- Let $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ be Pareto efficient. Then there is a an A-D equilibrium with price $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ that decentralizes the allocation $(\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$.

Sequential Markets Market Structure

- Now we will let trade take place sequentially in spot markets in each period, event-history pair.

- One period contingent IOU's: financial contracts bought in period t that pay out one unit of the consumption good in $t+1$ only for a particular realization of $s_{t+1} = j$ tomorrow.

- $Q_t(s^t, s_{t+1})$: price at period t of a contract that pays out one unit of consumption in period $t+1$ if and only if tomorrow's event is s_{t+1} (zero-coupon bonds).

- $a^i_{t+1}(s^t, s_{t+1})$: quantities of these Arrow securities bought (or sold) at period t by agent i.

- These contracts are often called Arrow securities, contingent claims or one-period insurance contracts.

Period-by-period Budget Constraint

- The period t, event history s^t budget constraint of agent i is given by

$$c_t^i(s^t) + \sum_{s_{t+1}|s^t} Q_t(s^t, s_{t+1}) a_{t+1}^i(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t)$$

- Note: we only have prices and quantities.

- Many economists use expectations. We will later see why. However, this is *bad* practice.

Natural Debt Limit

- We need to rule out Ponzi schemes.

- Tail of endowment distribution:

$$A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} \frac{p_\tau(s^\tau)}{p_t(s^t)} e_\tau^i(s^\tau)$$

- $A_t^i(s^t)$ is known as the *natural debt limit*.

- Then:

$$-a_{t+1}^i(s^{t+1}) \leq A_{t+1}^i(s^{t+1})$$

Sequential Markets Equilibrium

A SM equilibrium is prices for Arrow securities $\{\widehat{Q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$ and allocations $\left\{\left(\hat{c}_t^i(s^t), \left\{\hat{a}_{t+1}^i(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\right)_{i=1,..,I}\right\}_{t=0,s^t\in S^t}^{\infty}$ such that:

1. For $i = 1, .., I$, given $\{\widehat{Q}_t(s^t, s_{t+1})\}_{t=0,s^t\in S^t,s_{t+1}\in S}^{\infty}$, for all i,

   $\{\hat{c}_t^i(s^t), \left\{\hat{a}_{t+1}^i(s^t, s_{t+1})\right\}_{s_{t+1}\in S}\}_{t=0,s^t\in S^t}^{\infty}$ solves

$$\max_{\{c_t^i(s^t),\left\{a_{t+1}^i(s^t,s_{t+1})\right\}_{s_{t+1}\in S}\}_{t=0,s^t\in S^t}^{\infty}} \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi(s^t)u(c_t^i(s^t))$$

$$\text{s.t. } c_t^i(s^t) + \sum_{s_{t+1}|s^t}\widehat{Q}_t(s^t, s_{t+1})a_{t+1}^i(s^t, s_{t+1}) \leq e_t^i(s^t) + a_t^i(s^t)$$

$$c_t^i(s^t) \geq 0 \text{ for all } t, s^t \in S^t$$

$$a_{t+1}^i(s^t, s_{t+1}) \geq -A_{t+1}^i(s^{t+1}) \text{ for all } t, s^t \in S^t$$

2. For all $t \geq 0$

$$\begin{aligned}\sum_{i=1}^{I} \hat{c}_t^i(s^t) &= \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, s^t \in S^t \\ \sum_{i=1}^{I} \hat{a}_{t+1}^i(s^t, s_{t+1}) &= 0 \text{ for all } t, s^t \in S^t \text{ and all } s_{t+1} \in S\end{aligned}$$

Equivalence of Arrow-Debreu and Sequential Markets Equilibria

- A full set of one-period Arrow securities is sufficient to make markets "sequentially complete."

- Any (nonnegative) consumption allocation is attainable with an appropriate sequence of Arrow security holdings $\{a_{t+1}(s^t, s_{t+1})\}$ satisfying all sequential markets budget constraints.

- Later, when we talk about asset pricing, we will discuss how to use $Q_t(s^t, s_{t+1} = j)$ to price any other security.

Pareto Problem

- We will extensively exploit the two welfare theorems.

- Negishi's (1960) method to compute competitive equilibria:

  1. We fix some Pareto weights.
  2. We solve the Pareto problem associated to those weights.
  3. We decentralize the resulting allocation using the second welfare theorem.

- All competitive equilibria correspond to some Pareto weights.

Social Planner's Problem

- We solve the social planners problem:

$$\max_{(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}} \sum_{i=1}^{I} \alpha_i \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \beta^t \pi(s^t) u(c_t^i(s^t))$$

$$\text{s.t.} \quad \sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) \text{ for all } t, s^t \in S^t$$

$$c_t^i(s^t) \geq 0 \text{ for all } t, s^t \in S^t$$

where α_i are the Pareto weights.

- An allocation $(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ is Pareto efficient if and only if it solves the social planners problem for some $(\alpha_i)_{i=1,...,I} \in [0,1]$.

Lagrangian Formulation

- We write the lagrangian for the problem:

$$\max_{(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}} \sum_{t=0}^{\infty} \sum_{s^t\in S^t} \left\{ \begin{array}{c} \sum_{i=1}^{I} \alpha_i \beta^t \pi(s^t) u(c_t^i(s^t)) \\ +\lambda_t \left(s^t\right) \left(\sum_{i=1}^{I} \left[e_t^i(s^t) - c_t^i(s^t)\right]\right) \end{array} \right\}$$

where $\lambda_t \left(s^t\right)$ is the state-dependent lagrangian multiplier.

- We forget about the non-negativity constraints.

Perfect Insurance

- We take first order conditions in the previous problem:

$$\alpha_i \beta^t \pi(s^t) u'(c_t^i(s^t)) = \lambda_t \left(s^t\right) \text{ for all } i, t, s^t \in S^t$$

- Then, by dividing the condition for two different agents:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\alpha_j}{\alpha_i}$$

- An allocation $(\{c_t^i(s^t)\}_{t=0, s^t \in S^t}^{\infty})_{i=1,..,I}$ has perfect consumption insurance if the ratio of marginal utilities between two agents is constant across time and states.

Irrelevance of History

- From previous equation, and making $j = 1$:

$$c_t^i(s^t) = u'^{-1}\left(\frac{\alpha_1}{\alpha_i}\beta^t\pi(s^t)u'(c_t^1(s^t))\right)$$

- Summing over individuals and using aggregate resource constraint:

$$\sum_{i=1}^{I} e_t^i(s^t) = \sum_{i=1}^{I} u'^{-1}\left(\frac{\alpha_1}{\alpha_i}\beta^t\pi(s^t)u'(c_t^1(s^t))\right)$$

which is one equation on one unknown, $c_t^1(s^t)$.

- Then, the pareto-efficient allocation $(\{c_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty})_{i=1,..,I}$ only depends on aggregate endowment and not on s^t.

Theory Confronts Data

- Perfect insurance implies proportional changes in marginal utilities as a response to aggregate shocks.

- Do we see perfect risk-sharing in the data?

- Surprasingly more difficult to answer than you would think.

- Let us suppose we have CRRA utility function. Then, perfect insurance implies:

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\alpha_i}{\alpha_j}\right)^{\frac{1}{\gamma}}$$

Individual Consumption

- Now, $c_t^i(s^t) = \frac{c_t^j(s^t)}{\alpha_j^{\frac{1}{\gamma}}} \alpha_i^{\frac{1}{\gamma}}$ and we sum over i

$$\sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} e_t^i(s^t) = \frac{c_t^j(s^t)}{\alpha_j^{\frac{1}{\gamma}}} \sum_{i=1}^{I} \alpha_i^{\frac{1}{\gamma}}$$

- Then:

$$c_t^j(s^t) = \frac{\alpha_j^{\frac{1}{\gamma}}}{\sum_{i=1}^{I} \alpha_i^{\frac{1}{\gamma}}} \sum_{i=1}^{I} e_t^i(s^t) = \theta_j y_t(s^t)$$

i.e., each agent consumes a constant fraction of the aggregate endowment.

Individual Level Regressions

- Take logs:

$$\log c_t^j(s^t) = \log \theta_j + \log y_t(s^t)$$

- If we take first differences,

$$\Delta \log c_t^j(s^t) = \Delta \log y_t(s^t)$$

- Equation we can estimate:

$$\Delta \log c_t^j(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e_t^i(s^t) + \varepsilon_t^i$$

Estimating the Equation

- How do we estimate?

$$\Delta \log c_t^j(s^t) = \alpha_1 \Delta \log y_t(s^t) + \alpha_2 \Delta \log e_t^i(s^t) + \varepsilon_t^i$$

- CEX data.

- We get α_2 is different from zero (despite measurement error).

- Excess sensitivity of consumption by another name!

- Possible explanation?

Permanent Income Hypothesis

- Build the Lagrangian of the problem of the household i:

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi(s^t)u(c_t^i(s^t)) - \mu_i\sum_{t=0}^{\infty}\sum_{s^t\in S^t}p_t(s^t)\left(e_t^i(s^t) - c_t^i(s^t)\right)$$

Note: we have only one multiplier μ_i.

- Then, first order conditions are

$$\beta^t\pi(s^t)u'(c_t^i(s^t)) = \mu_i p_t(s^t)\text{for all } t, s^t \in S^t$$

- Substituting into the budget constraint:

$$\sum_{t=0}^{\infty}\sum_{s^t\in S^t}\frac{1}{\mu_i}\beta^t\pi(s^t)u'(c_t^i(s^t))\left(e_t^i(s^t) - c_t^i(s^t)\right) = 0$$

No Aggregate Shocks

- Assume that $\sum_{i=1}^{I} e_t^i(s^t)$ is constant over time. From perfect insurance, we know then that $c_t^i(s^t)$ is also constant. Let's call it $\widehat{c}^i$.

- Then (cancelling constants)

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) \left(e_t^i(s^t) - \widehat{c}^i \right) = 0 \Rightarrow$$

$$\widehat{c}^i = (1-\beta) \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) e_t^i(s^t)$$

- Later, we will see that with no aggregate shocks, $\beta^{-1} = 1 + r$.

- Then,

$$\widehat{c}^i = \frac{r}{1+r} \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \left(\frac{1}{1+r} \right)^t \pi(s^t) e_t^i(s^t)$$

Milton Friedman's permanent income model.