# Introduction to Uncertainty

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## Introduction to Uncertainty

- Modern macro studies stochastic processes of observed variables.

- Two elements:

  1. Dynamics.

  2. Uncertainty.

- We will introduce some basic concepts by presenting a pure exchange economy with stochastic endowments.

- In this lecture, we will present the expected discounted utility and we will use it to assess the welfare cost of the business cycle.

## Time

- Discrete time $t \in \{0, 1, 2, ...\}$.

- Why discrete time?

  1. Economic data is discrete.

  2. Easier math.

- Comparison with continuous time:

  1. Discretize observables.

  2. More involved math (stochastic calculus) but often we have extremely powerful results.

- Calendar versus planning time.

# Events

- One event $s_t$ happens in each period.

- $s_t \in S = \{1, 2, ..., N\}$ .

- Note:

  1. $S$ is a finite set. We will later talk about measure theory.

  2. $S$ does not depend on time.

- Event history $s^t = (s_0, s_1, ..., s_t) \in S \times ... \times S = S^{t+1}$.

# Probabilities

- Probability of $s^t$ is $\pi(s^t)$.
- Conditional probability of $s_{t+1}$ is $\pi\left(s_{t+1}|\, s^t\right)$.
- At this moment, we are not imposing any transition probability among states across time.
- Our notation allows the *particular* cases:

$$\begin{aligned}\pi\left(s_{t+1}|\, s^t\right) &= \pi\left(s_{t+1}\right)\\ \pi\left(s_{t+1}|\, s^t\right) &= \pi\left(s_{t+1}|\, s_t\right)\end{aligned}$$

## Commodity Space

- One good in the economy.

- However, good indexed by event history over infinite time. Hence our commodity space is slightly more complicated (see chapter 15 in SLP).

- Commodity space: $(C, \|\cdot\|)$.

- We pick $l_\infty$, i.e., the space of sequences $c = (c_0, c_1, ...), c_n \in \mathbb{R}$ that are bounded in the norm:

$$\|c\|_\infty = \sup_i |c_i|$$

## Household Preferences

- Preferences admit a representation:

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t))$$

- This is known as the von Neumann-Morgenstern expected utility function.

- Remember:

  1. Key assumptions: continuity and independence axioms.

  2. Linear in probabilities.

  3. Cardinal utility: unique only up to an affine transformation.

## Facts about Utility Function I: Time Separability

- Total utility $c$ equals the expected discounted sum of period (or instantaneous) utility $u(c_t(s^t))$.

- The period utility at time $t$ only depends on consumption in period $t$ and not on consumption in other periods.

- This formulation rules out, among other things, habit persistence.

- However, it is easy to relax: recursive utility functions.

## Facts about Utility Function II: Time Discounting

- $\beta < 1$ indicates that agents are impatient.

- $\beta$ is called the (subjective) time discount factor.

- The subjective time discount rate $\rho$ is defined by $\beta = \frac{1}{1+\rho}$.

- Assumption: constant over time$\rightarrow$exponential discounting.

- Alternatives: hyperbolic discounting.

## Facts about Utility Function III: Risk Aversion

- Arrow-Pratt Absolute Risk Aversion:

$$ARA = -\frac{u''(c)}{u'(c)}$$

Why do we divide by $u'(c)$?

- Arrow-Pratt Relative Risk Aversion:

$$RRA = -\frac{u''(c)}{u'(c)}c$$

Interpretation.

# Common Utility Functions

- Constant Absolute Risk Aversion (CARA):

$$-e^{-ac}$$

- Constant Relative Risk Aversion (CRRA):

$$\begin{aligned} \frac{c^{1-\gamma}-1}{1-\gamma} \text{ for } \gamma &\neq 1 \\ \log c \text{ for } \gamma &= 1 \end{aligned}$$

(you need to take limits and apply L'Hôpital's rule).

# CRRA Utility Functions

- $\gamma$ plays a dual role controlling risk-aversion and intertemporal substitution.

- Coefficient of Relative Risk-aversion:

$$-\frac{u''(c)}{u'(c)}c = \gamma$$

- Elasticity of Intertemporal Substitution:

$$-\frac{u(c_2)/u(c_1)}{c_2/c_1}\frac{d\left(c_2/c_1\right)}{d\left(u(c_2)/u(c_1)\right)} = \frac{1}{\gamma}$$

- Advantages and disadvantages.

## Why CRRA Utility Functions?

- Market price of risk has been roughly constant over the last two centuries.

- This observation suggests that risk aversion should be relatively constant over the wealth levels.

# Cost of Business Cycles

- Simple CRRA utility function already answers many questions.

- Lucas (1987), *Models of Business Cycles*: What is the welfare cost of business cycles?

- Importance of question:

  1. Limits of stabilization policy.

  2. Macroeconomic priorities.

## A Process for Consumption

- Assume that consumption evolves over time as:

$$c_t = \mu^t (1+\lambda) e^{-\frac{1}{2}\sigma_z^2} z_t c$$

where $\log z_t \sim \mathcal{N}\left(0, \sigma_z^2\right)$.

- The moment generating function of a lognormal distribution implies:

$$\mathbb{E}\left(z_t^m\right) = e^{\frac{m^2\sigma_z^2}{2}}$$

- Then:

$$\mathbb{E}\left(e^{-\frac{1}{2}\sigma_z^2} z_t\right) = 1$$

$$\mathbb{E}\left(z_t^{1-\sigma}\right) = e^{\frac{1}{2}(1-\gamma)^2\sigma_z^2}$$

## A Compensating Differential

- We want to find the value of $\lambda$ such that:

$$\mathbb{E}\frac{c_t^{1-\gamma}-1}{1-\gamma}=\frac{\left(\mu^t c\right)^{1-\gamma}-1}{1-\gamma}$$

- If these is true period by period and event by event, it should also be true when we sum up.

- Moreover, the converse is also true: $\lambda$ is the smallest number that makes total utilities over time to be equal. Why? Because of the CRRA and the i.i.d. structure of $z_t$.

- Interpretation: $\lambda$ is the welfare cost of uncertainty, i.e., by how much we need to raise consumption in every period and state.

## Finding $\lambda$

- Dropping irrelevant constants, $\lambda$ solves:

$$\mathbb{E}\left(1+\lambda\right)\left(e^{-\frac{1}{2}\sigma_z^2}z_t\right)^{1-\gamma} = 1 \Rightarrow$$
$$\left(1+\lambda\right)e^{-\frac{1}{2}\sigma_z^2(1-\gamma)}\mathbb{E}z_t^{1-\gamma} = 1 \Rightarrow$$
$$\left(1+\lambda\right)e^{-\frac{1}{2}\sigma_z^2(1-\gamma)+\frac{1}{2}(1-\gamma)^2\sigma_z^2} = 1 \Rightarrow$$
$$\left(1+\lambda\right)e^{-\frac{1}{2}\gamma\sigma_z^2} = 1$$

- Taking logs: $\lambda \approx \frac{1}{2}\gamma\sigma_z^2$.

- Let us put some numbers here. Using quarterly U.S. data 1947-2006, $\sigma_z^2 = (0.033)^2$. What is $\gamma$?

## Size of $\gamma$

- Most evidence suggests that $\gamma$ is low, between 1 and 3. At most 10.

- Types of evidence:

  1. Questionnaires.

  2. Experiments.

  3. Econometric estimates from observed behavior.

- Two powerful arguments from growth theory international comparisons. We will revisit these points when we talk about asset pricing.

- Rabin's (2000): "Risk Aversion and Expected-Utility Theory: A Calibration Theorem.", *Econometrica*.

## An Estimate of the Cost of the Business Cycle

- Let us take $\gamma = 1$ as a benchmark number. Then, we have:

$$\lambda \approx \frac{1}{2}\gamma\sigma_z^2 = \frac{1}{2} * 1 * (0.033)^2 = 0.0005$$

- Even if we take $\gamma = 10$ as an upper bound:

$$\lambda \approx \frac{1}{2}\gamma\sigma_z^2 = \frac{1}{2} * 10 * (0.033)^2 = 0.005$$

- These are extremely small numbers.

- Later we will see how this finding is intimately linked with the Equity premium puzzle.

- How could we turn around this result?

# Alternatives Routes

- We assumed:

  1. Representative agent.

  2. Exogenous lognormal consumption.

  3. Expected utility.

- How important are each of these three assumptions?

## Representative Agent

- Representative agent: fluctuations are at the margin.

- Lucas is very explicit about the possible costs of inequality.

- We will see in the next lecture that with complete markets we will have perfect risk sharing.

- But the interesting question is the effects of business cycles with incomplete markets and heterogeneity.

- Krusell and Smith (2002), loss of 0.001 of average consumption, 65% of households *lose* when business cycles are removed.

## Exogenous Lognormal Consumption

- A combined hypothesis: exogenous consumption+lognormal consumption.

- Exogenous consumption⇒Cho and Cooley (2001), business cycles *may* increase welfare: mean versus spread effect.

- Lognormal consumption⇒great depressions? Chaterjee and Corbae (2005): welfare cost of 0.0187. They calibrate a great depression every 87 years.

- A nonparametric approach by Álvarez and Jermann (2004) suggests costs between 0.0008 and 0.0049.

## Problems of Expected Utility

- We have representation:

$$U(c) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi(s^t) u(c_t(s^t))$$

- Three strong assumptions:

  1. Intertemporal elasticity of substitution and risk aversion are determined by just one parameter.

  2. Temporal separability.

  3. Expected utility.

- All are problematic and they may affect our calculations.

## Recursive Utility

- Espstein-Zin preferences (1989):

$$U_t = \left[(1-\beta)\, c_t^{\rho} + \beta \left(\mathbb{E}_t U_{t+1}^{\alpha}\right)^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}}$$

  separates elasticity of substitution:

$$\gamma = \frac{1}{1-\rho}$$

  from risk-aversion $\alpha$.

- Applied to evaluate cost of business cycles by Tallirini (2000).

## Risk in the Long Run

- Bansal and Yaron (2004): difficult to distinguish a long run component from a random walk.

- Implications for the equity premium.

- Croce (2006): cost of business cycle.

## Temporal Anomalies

- Present-bias. Frederich, Lowenstein, and O'Donoghue (2002), "Time Discounting and Time Preference: A Critical Review". *Journal of Economic Literature*.

- Explanations:

  1. Hyperbolic discounting (Phelps and Pollack, 1968, Laibson, 1996):

  $$\sum_{t=0}^{\infty} \delta \beta^t u(c_t)$$

  2. Temptation: Gul and Pesendorfer (2003).

# Uncertainty Anomalies

1. Framing effects (Kahneman and Tversky).

2. Allais paradox. Three prizes in a lottery: $\{0, 1, 10\}$

   Problem 1: $L_1 = (0, 1, 0)$ versus $L_2 = (0.01, 0.89, 0.1)$.

   Problem 2: $L_3 = (0.89, 0.11, 0)$ versus $L_4 = (0.9, 0, 0.1)$.

3. Ellsberg paradox.

## Ambiguity Aversion

- Knight (1921) risk versus uncertainty.

- Gilboa and Schmidler (1989):

$$\min_{Q\in\mathcal{P}} \mathbb{E}_Q u(c)$$

- Two possible extensions:

  1. Choice over time.

  2. General class of ambiguity aversion.

# Choice over Time

- Epstein and Schneider (2003):

$$\min_{Q \in \mathcal{P}} \mathbb{E}_Q \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Difficult technical assumption⇒rectangularity.

## Ambiguity and the Variational Representation of Preferences

- Maccheroni, Marinacci, and Rustichini (2006):

$$\min_{Q \in \mathcal{P}} \left\{ \mathbb{E}_Q u(c) + \phi(Q) \right\}$$

- The function $u$ represents risk attitudes while the index $c$ captures ambiguities attitudes.

- They extend it to the intertemporal case.

- One particular example:

$$\min_{Q \in \mathcal{P}} \left\{ \mathbb{E}_Q u(c) + \theta R(Q \| P) \right\}$$

- Hansen and Sargent's (2006) research program on robust control.