Toward an Efficiency Rationale for the Public Provision of Private Goods*

Hanming Fang† Peter Norman‡

July 16, 2009

Abstract

We show that public provision of private goods by local governments may be justified on pure efficiency grounds. A local government's involvement in the provision of the private good supplies it with information about individuals' private good purchases that facilitates more efficient revenue extraction for the provision of public goods. Public provision of the private good improves economic efficiency under a condition that is always fulfilled under stochastic independence and satisfied for an open set of joint distributions. Our theory predicts that only private goods for which consumers cannot price arbitrage can be used by the government to improve efficiency.

Keywords: Publicly Provided Private Goods; Constrained Efficiency; Optimal Taxation.

JEL Classification Number: D61, D82, H42.

---

*We thank Bill Zame and many seminar and conference participants for helpful comments. All remaining errors are ours.

†Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia, PA 19104. Email: hanming.fang@econ.upenn.edu

‡Department of Economics, University of North Carolina at Chapel Hill, 107 Gardner Hall, CB 3305, Chapel Hill, NC 27599-3305. Email: normanp@email.unc.edu

1 Introduction

Governments at all levels not only provide public goods, but also devote considerable resources to the provision of private goods such as health care, housing and education. This is puzzling in the light of the standard public finance justifications for government intervention in the marketplace. Over the last twenty years a number of models have been developed to rationalize public provision of private goods; but almost all such models ultimately explain it as an *instrument to redistribute* from the wealthy to the poor. However, some publicly provided private goods have arguably either neutral or regressive distributional consequences, with the publicly-provided higher education being one of the prime examples.

This paper provides a novel explanation for public provision of private goods. We adapt familiar ideas from the literature on commodity bundling to a welfare optimization problem involving both public and private goods, and show that public provision of private goods can be explained as an *instrument to improve economic efficiency.* Using publicly-provided higher education as a concrete example, our model implies that, in an efficient allocation, the in-state tuition for college education (a private good) should be below the out-of-state tuition to provide incentives for the citizens (i.e. parents of current and/or future college students) to stay in the State and pay the taxes that finance the local public goods.

We model the government as a benevolent social planner operating in an economy with both private and excludable public goods. A consumer's preference toward the public and private goods is her private information and unobserved by the planner. We also assume that there are no complementarities between the private and the public goods on either the production or the consumption side. As a benchmark, we consider the case where *by assumption* the public good and the private good are provided separately. That is, each good is allocated without use of individual data from the other market, thus ruling out mechanisms, for example, where the price charged for the private good depends on whether the individual is willing to pay the user fee for the public good. Cross subsidies, however, are allowed. Alternatively, one can think of the separation provision problem as one where two independent government agencies are separately in charge of the public and private good provisions, but where the government as a whole faces a single integrated budget constraint. Besides the restriction that information cannot be shared between the two agencies, we impose no other restrictions on the feasible mechanisms. However, using arguments from the optimal auction literature we provide a very simple characterization of the best separate provision mechanism, which can be described as an access fee for the public good, a provision probability of the public good, and a price for the private good. Thus the best separate provision mechanism is fully consistent with a decentralized market for the private good as all the agency in charge of the private good market needs to do is to set a constant sales tax for the private good.

We contrast the best separate provision mechanism with the case where the goods are jointly provided. By "joint provision," we mean that *all information* can be used for decisions on pricing and provision of *both* goods. This leads to a model where types are multidimensional, thus preventing us from characterizing the constrained optimal mechanism due to well-understood technical difficulties. Fortunately, our primary interest is the qualitative question of *when public provision of the private good is welfare improving.* For this, it is sufficient to demonstrate that there is *some joint provision mechanism that outperforms the best separate provision mechanism.* We thus consider small deviations from the best separate provision mechanism and furthermore restrict ourselves to deviations characterized by three fixed prices, one for the private good only, one for the public good only, and one for the bundle consisting of both goods. Under a condition, which is satisfied by a large set of joint distributions, including the case where valuations are independent, there indeed exists deviation mechanisms within the restricted class that improve upon the best separate provision mechanism. The welfare improving mechanism we identify is generally not constrained optimal because we are making a local argument using only a particular class of pricing mechanisms, but the constrained optimal mechanism must weakly outperform the simple mechanism under our consideration. Hence the optimal joint provision mechanism *must condition prices for private good consumption on valuations for the public good.* This *cannot be implemented using an anonymous market mechanism for the private good*, which is why we interpret non-separability between the markets as public provision.

Our paper makes three main contributions to the literature on the public provision of private goods.[1] First, in our paper the driving force for the welfare improvement from the public provision of private goods is that it generates information that allows for more efficient revenue extraction which can then be used to fund typically under-provided public good. In contrast, all existing explanations for the public provision of private goods rely upon preferences for redistribution. Since many publicly-provided private goods indeed have neutral or regressive distributional effects, our efficiency based explanation complements the redistribution based existing literature.

Second, in this paper we adopt a mechanism design approach, which forces us to establish the efficiency property of the public provision of private goods by comparing the welfare under some joint provision mechanism with the *best* outcome that can be implemented under separate provisions. This is in stark contrast to most of the existing literature where the benchmark outcome is often derived under various implicit restrictions on the operation of the private good market.

Third, our paper not only explains that public provision of some private goods may improve efficiency, but also highlights that *only private goods for which consumers can not engage in price arbitrage* could be used by the government to improve efficiency. This natural boundary on the public provision of private goods is desirable from a theoretical viewpoint, and is in contrast with the existing redistribution-based literature where it is silent on which private goods should be used as the instrument for redistribution. The no arbitrage restriction also seems realistic for many goods that are publicly provided in the real world. For example, the typical publicly-provided private goods, public schools (including public colleges) and public health insurance, and public health care, are all commodities that are difficult to resell.

Our analysis also illustrates a more general point related to public finance. Since Ramsey (1927), it has been standard in the optimal taxation literature to ask how to best raise a given

[1] See Section 5 for a more detailed review of the literature.

target revenue without considering what the tax revenue is intended for. Similarly, public good provision and other public expenditure problems always take the size of any outside funds as given. This paper provides an example where the taxation and expenditure problems are non-separable: the optimal commodity tax to finance the public good should depend on whether the consumer gets access to the public good.[2] Hence, the analytically convenient dichotomy between government expenditures and revenue may result in an efficiency loss.

The remainder of the paper is structured as follows. Section 2 presents the model; Section 3 characterizes the best separate provision mechanism; Section 4 presents our main results regarding the optimality of public provision of private goods; Section 5 reviews the related literature in greater details and highlights the contributions of this paper; and finally Section 6 concludes.

2 The Environment

An economy is populated by a continuum of *ex ante* identical agents. Agents have preferences over a binary and excludable public good, a binary private good and a perfectly divisible numeraire good that we call "money." The public good can be produced at a per capita cost $K > 0$; and the binary private good can be produced at unit cost $c > 0$.

An agent is characterized by her type $\boldsymbol{\theta} \equiv (\theta_G, \theta_P) \in \boldsymbol{\Theta} \equiv \Theta_G \times \Theta_P = \left[\underline{\theta}_G, \overline{\theta}_G\right] \times \left[\underline{\theta}_P, \overline{\theta}_P\right]$ where θ_G is her valuation for the public good and θ_P is her valuation for the private good, and these valuations are her private information. To avoid trivialities, we assume that $0 \leq c < \overline{\theta}_P$ and $0 < K < \overline{\theta}_G$. We denote $H : \Theta_G \times \Theta_P \to [0, 1]$ as the cumulative distribution function of agent types (θ_G, θ_P), and write $H_G : \Theta_G \to [0, 1]$ and $H_P : \Theta_P \to [0, 1]$ the respective marginal cumulative distributions of θ_G and θ_P. Agents are risk neutral and have expected payoff function

$$u\left(\phi_G, \phi_P, m; \theta_G, \theta_P\right) = \phi_G \theta_G + \phi_P \theta_P - m, \tag{1}$$

where ϕ_G is the probability of consuming the public good, ϕ_P is the probability of consuming the private good and m is the transfer of money.

3 Separate Provision Mechanisms

In this section we analyze a benchmark problem where the planner cannot condition either the price or the access rule for one good on a consumer's reported valuation for the other good. Formally, a *separate provision mechanism* is an object $(\pi, \phi_G, t_G, \phi_P, t_P)$, where $\pi \in [0, 1]$ is the probability that the public good is provided; $\phi_G : \Theta_G \to [0, 1]$ is the probability of access to the public good; $t_G : \Theta_G \to \mathrm{R}$ is the fee for consuming the public good; $\phi_P : \Theta_P \to [0, 1]$ is the probability of consuming the private good; and $t_P : \Theta_P \to \mathrm{R}$ is the fee for consuming the private good.

[2] Related points are made in Boadway *et al.* (1998) and Blomquist and Christiansen (2007).

The restriction to separate provision mechanisms implies that (ϕ_G, t_G) are functions of θ_G only and *cannot depend on* θ_P, and that (ϕ_P, t_P) are functions of θ_P only and *cannot depend on* θ_G. One interpretation for the separate provision mechanism is that the markets are physically separated in space, and that the government agencies lack the technology to track the behavior of individual agents across markets. Alternatively, the constraint may simply reflect a desire not to intervene in the market for private goods beyond imposing a sales tax.

Note that in our definition of the separate provision mechanism, the provision probability π is a scalar and that $(\phi_G, t_G, \phi_P, t_P)$ depend on a single agent's type realization. This simplification is a result of modelling the agents in the economy as a continuum. This simplifies the analysis tremendously relative to the finite model, where all decision rules depend on the full type profile. However, in a finite economy with sufficiently many agents, the effect of making the provision rule depend on others' announcements becomes negligible due to intuition similar to that of the "Paradox of Voting." As a result, the continuum model can be seen as an approximation of a finite large economy in our set up (see, e.g., Schmitz 1997, Hellwig 2003, Norman 2004, and Fang and Norman 2009).[3, 4]

3.1 The Problem

Given a separate provision mechanism $(\pi, \phi_G, t_G, \phi_P, t_P)$, the *ex ante* expected average utility of all the agents in the economy is:

$$\begin{aligned} & \int_{\Theta_G} \int_{\Theta_P} \left[\phi_G(\theta_G)\theta_G - t_G(\theta_G) + \phi_P(\theta_P)\theta_P - t_P(\theta_P)\right] dH(\theta_G, \theta_P) \\ = \quad & \sum_{J=G,P} \int_{\Theta_J} \left[\phi_J(\theta_J)\theta_J - t_J(\theta_J)\right] dH_J(\theta_J). \end{aligned} \tag{2}$$

Equation (2) illustrates the simple but important fact that the welfare criterion under the separate provision mechanism is a sum of two terms that separate completely across the two markets. The

[3] This is not always the case for mechanism design problems. For example, Groves and other pivot mechanisms can not even be formulated when there is a continuum of agents, because it is impossible to make individual agents pivotal while at the same time maintaining measurability.

[4] However, the continuum specification has an awkward technical aspect. We treat the distribution H as a cross-section of agents, which is necessary to interpret the right hand side of the balanced budget constraint (3d) in Subsection 3.1 as the total revenue from the mechanism. At the same time, the distribution H is also interpreted as the *ex ante* probability distribution over types for any individual agent. As is well-known, this is inconsistent with stochastic independence across agents in standard probability measures (see, however, Al-Najjar 2004 for a solution to the problem).

best separate provision mechanism thus must solve the following problem:

$$\max_{\left\{\pi,(\phi_J,t_J)_{J\in\{P,G\}}\right\}} \sum_{J\in\{G,P\}} \int_{\Theta_J} \left[\phi_J(\theta_J)\theta_J - t_J(\theta_J)\right] dH_J(\theta_J) \tag{3a}$$

$$\text{s.t.} \quad \sum_{J\in\{G,P\}} \left[\phi_J(\theta_J)\theta_J - t_J(\theta_J)\right] \geq \sum_{J\in\{G,P\}} \left[\phi_J(\widehat{\theta}_J)\theta_J - t_J(\widehat{\theta}_J)\right] \qquad \forall \boldsymbol{\theta}, \widehat{\boldsymbol{\theta}} \in \boldsymbol{\Theta} \tag{3b}$$

$$\sum_{J\in\{G,P\}} \left[\phi_J(\theta_J)\theta_J - t_J(\theta_J)\right] \geq 0 \qquad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta} \tag{3c}$$

$$K\pi + \int_{\Theta_P} c\phi_P(\theta) dH_P(\theta_P) \leq \int_{\Theta_J} t_J(\theta_J) dH_J(\theta_J) \tag{3d}$$

$$0 \leq \phi_G(\theta_G) \leq \pi \qquad \forall \theta_G \in \Theta_G, \tag{3e}$$

where (3b) represents the incentive compatibility constraints for truth-telling; (3c) represents the participation constraints to ensure that no agent is made strictly worse off than opting out by, e.g., moving to a different jurisdiction;[5] (3d) is the balanced budget constraint so that the total costs for the production of the public and the private goods do not exceed the total revenue collected from the agents;[6, 7] and finally, constraint (3e) restricts the probability of accessing the public good to be no more than the probability that the public good is provided.

3.2 The Characterization

Note that an important implication of the restriction to separate provision mechanisms is that the incentive compatibility constraints (3b) hold if and only if, for $J \in \{G, P\}$,

$$\phi_J(\theta_J)\theta_J - t_J(\theta_J) \geq \phi_J(\widehat{\theta}_J)\theta_J - t_J(\widehat{\theta}_J), \quad \forall \theta_J, \widehat{\theta}_J \in \Theta_J. \tag{4}$$

That is, incentive compatibility separates into "public good incentive compatibility" and "private good incentive compatibility."

More importantly, it can be shown that the maximization problem (3) can be decomposed into two separate uni-dimensional mechanism design problems. The two separate uni-dimensional problems have the familiar form as in Myerson (1981) and involve a term that has been known as "virtual valuation," which is defined as

$$x_J(\theta_J) \equiv \theta_J - \frac{1 - H_J(\theta_J)}{h_J(\theta_J)}. \tag{5}$$

[5] The imposition of the participation constraint means that our theory is suited to explain the public provision of private goods by *local* governments. Federal government can typically rely on coercion and does not have to respect individuals' participation constraints. Without the participation constraint, it should be clear that the first best can be achieved by simply combining a mechanism a la d'Aspremont and Gérard-Varet (1979) for the public good with marginal cost pricing for the private good (see, e.g., Mailath and Postlewaite 1990).

[6] Note that the total cost depends on the probability of provision π, but not on the probability of consumption $\phi_G(\cdot)$. This reflects the assumption that the good is non-rival.

[7] Also note that (3d) only requires that the resources are balanced in expectation, but this is without loss of generality because one can easily adjust transfers without changing the interim expected payoffs in such a way as to balance the budget *ex post* for sure (see, e.g., Borgers and Norman 2008).

Specifically, let $(\pi^*, \phi_G^*, t_G^*, \phi_P^*, t_P^*)$ be a solution to problem (3); the two uni-dimensional problems can be defined as follows. The first is an excludable public good provision problem taking (ϕ_P^*, t_P^*) as given:

$$\max_{\{\pi,\phi_G\}} \int_{\Theta_G} \left[\phi_G(\theta_G)\,\theta_G\right] dH_G(\theta_P) - K\pi \tag{6a}$$

$$\text{s.t. } 0 \leq \int_{\Theta_G} \phi_G(\theta_G)\, x_G(\theta_G)\, dH_G(\theta_G) - K\pi + \int_{\Theta_P} \left[t_P^*(\theta_P) - c\phi_P^*(\theta_P)\right] dH_P(\theta_P) \tag{6b}$$

$$0 \leq \phi_G(\theta_G) \leq \pi \text{ for all } \theta_G \tag{6c}$$

$$\phi_G(\cdot) \text{ is weakly increasing.} \tag{6d}$$

And the second is a private good allocation problem taking (π^*, ϕ_G^*, t_G^*) as given:

$$\max_{\{\phi_P\}} \int_{\Theta_P} \left[\phi_P(\theta_P)(\theta_P - c)\right] dH_P(\theta_P) \tag{7a}$$

$$\text{s.t. } 0 \leq \int_{\Theta_P} \phi_P(\theta_P)\, x_P(\theta_P)\, dH_P(\theta_P) + \int_{\Theta_G} t_G^*(\theta_G)\, dH_G(\theta_G) - K\pi^* \tag{7b}$$

$$0 \leq \phi_P(\theta_P) \leq 1 \text{ for all } \theta_P \tag{7c}$$

$$\theta_P(\cdot) \text{ is weakly increasing.} \tag{7d}$$

The following lemma, whose detailed derivation is in the appendix, provides the link between the original planning problem (3) and the two separate uni-dimensional problems (6) and (7):

Lemma 1 *If $(\pi^*, \phi_G^*, t_G^*, \phi_P^*, t_P^*)$ solves the planning problem (3), then (π^*, ϕ_G^*) solves (6) and ϕ_P^* solves (7); moreover, for $J \in \{G, B\}$, $t_J^*(\theta_J) = \theta_J \phi_J^*(\theta_J) - \int_{\underline{\theta}_J}^{\theta_J} \phi_J^*(x)\, dx$ for all $\theta_J \in \Theta_J$.*

To facilitate further interpretations, observe that the problem for a *profit maximizing monopolist* for the public good would be to maximize

$$\int_{\Theta_G} \phi_G(\theta_G)\, x_G(\theta_G)\, dH_G(\theta_G) - K\pi \tag{8}$$

subject only to the constraints (6c) and (6d). For this problem, the "no-haggling" logic of Stokey (1979), Myerson (1981) and Riley and Zeckhauser (1983) immediately implies that the profit-maximizing mechanism is, without loss of generality, one where the monopolist charges a single price. However, this result does not extend to our problem where profits appear as a constraint in (6b). In general, the solution to the problem (6), for example, may very well be a randomized mechanism.[8] To rule out randomization in the optimum, we will assume that the "virtual valuation" $x_J(\theta_J)$ as defined in (5) is weakly increasing in θ_J, which is a standard regularity condition. Then

Lemma 2 *Suppose that $x_J(\theta_J)$ as defined in (5) is weakly increasing in θ_J for $J \in \{G, P\}$.*

1. If (π^*, ϕ_G^*) is a solution to (6), there exists some f such that

$$\phi_G^*(\theta_G) = \begin{cases} 0 & \text{if } \pi^*\theta_G < f \\ \pi^* & \text{if } \pi^*\theta_G \geq f; \end{cases} \tag{9}$$

[8] As a simple illustration, consider a case where there are two valuation types for the public good, θ_G^l and θ_G^h. Suppose that charging a flat fee equal to θ_G^l would violate the budget constraint, whereas charging θ_G^h would give a strict budget surplus. It is then obvious that welfare can be made higher by letting the low type agents consume the public good with some probability. The example can easily be extended to continuous densities.

2. If ϕ_P^* is a solution to (7), there exists p such that

$$\phi_P^*(\theta_P) = \begin{cases} 0 & \text{if } \theta_P < p \\ 1 & \text{if } \theta_P \geq p. \end{cases} \tag{10}$$

The proof relies on on the fact that $\phi_J(\theta_J)$ appear linearly in both the objective functions and the constraints. As the argument is quite standard, we here only sketch the proof for Part 1. We take as granted that the problem can be solved by Lagrangian techniques (see Hellwig 2005 for justification). Let $\lambda \geq 0$ be the multiplier on the integral constraint (6b), $\gamma(\theta_G)$ be the multiplier on constraint $\phi_G(\theta_G) \geq 0$, and $\omega(\theta_G)$ be the multiplier on constraint $\pi - \phi(\theta_G) \geq 0$. The optimality condition for $\phi(\theta_G)$ then reads:

$$\theta_G + \lambda x_G(\theta_G) + \frac{\gamma(\theta_G) - \omega(\theta_G)}{h_G(\theta_G)} = 0, \tag{11}$$

together with the appropriate complementary slackness conditions for the non-negativity constraints. As $\theta_G + \lambda x_G(\theta_G)$ is continuous and strictly increasing under the regularity condition that $x_G(\cdot)$ is weakly increasing in θ_G, it follows from (11) that the solution has a threshold property, i.e., there exists some θ_G^* such that $\phi_G^*(\theta_G) = 0$ if $\theta_G < \theta_G^*$, and $\phi_G^*(\theta_G) = \pi^*$ if $\theta_G \geq \theta_G^*$, exactly as described by (9) if we let $f = \theta_G^*/\pi^*$. Moreover, Lemma 1 implies that the threshold f in (9) is in fact closely associated transfer $t_G^*(\theta_G)$:

$$t_G^*(\theta_G) = \theta_G \phi_G^*(\theta_G) - \int_{\underline{\theta}_G}^{\theta_G} \phi_G^*(x)\, dx = \begin{cases} 0 & \text{if } \pi^*\theta_G < f \\ f & \text{if } \pi^*\theta_G \geq f. \end{cases}$$

Thus, the threshold f can be interpreted as an access fee for a lottery where consumers who pay f get access to the public good if it is provided, which happens with probability π^*.[9] The intuition for the proof of Part 2 is almost identical, with the threshold p interpreted as the price for the private good.

Lemmas 1 and 2 imply that, given the regularity conditions imposed, a solution to (3) may be obtained by solving the following simplified planning problem:

$$\max_{\{\pi, f, p\}} \pi \left[\int_{\frac{f}{\pi}}^{\overline{\theta}_G} \theta_G dH_G(\theta_G) - K \right] + \int_p^{\overline{\theta}_P} (\theta_P - c)\, dH_P(\theta_P) \tag{12}$$

$$\text{s.t} \qquad 0 \leq f \left[1 - H_G\left(\frac{f}{\pi} \right) \right] - \pi K + (p - c)\left[1 - H_P(p) \right], \tag{13}$$

$$0 \leq \pi \leq 1. \tag{14}$$

Proposition 1 *Suppose that* $\mathrm{E}(\theta_G | \theta_G \geq 0) > K$ *and that* $\underline{\theta}_G < K$. *Then, in any optimal solution* (π^*, f^*, p^*) *to (12): (1)* $p^* > 0$*; (2)* $\pi^* > 0$*; (3)* $f^* > 0$.

Proof. First, write $\chi \equiv \frac{f}{\pi}$ and let λ, μ and γ respectively be the Lagrangian multiplier for the constraint (13) and the boundary constraints $\pi \geq 0$ and $\pi \leq 1$. The Kuhn-Tucker necessary

[9] Alternatively, the fee could be charged only when the good is provided, in which case the relevant access price would be $\frac{f}{\pi^*}$.

conditions for an optimum are:

$$0 = \int_{\chi}^{\overline{\theta}_G} \theta_G dH_G(\theta_G) + (1+\lambda)\left[h_G(\chi)\chi^2 - K\right] + \mu - \gamma \tag{15}$$

$$0 = -\chi h_G(\chi) + \lambda\left[1 - H_G(\chi) - h_G(\chi)\chi\right] \tag{16}$$

$$0 = -(p-c)h_P(p) + \lambda\left[(1 - H_P(p)) - (p-c)h_P(p)\right] \tag{17}$$

$$0 = \lambda\left\{\pi\chi\left[1 - H_G(\chi)\right] + (p-c)\left[1 - H_P(p)\right] - K\pi\right\}, \lambda \geq 0 \tag{18}$$

$$\mu\pi = 0, \gamma(1-\pi) = 0, \mu \geq 0, \gamma \geq 0 \tag{19}$$

(Part 1): If $p^* < c$, then the first term on the right hand side in (17) is strictly positive and the second is weakly positive; thus the condition cannot hold. Suppose that $p^* = c$. Then, from (17) either $\lambda = 0$ or $1 - H_P(c) = 0$. Since the second condition is ruled out by the assumption that $c < \overline{\theta}_P$, the only possibility that remains is that $\lambda = 0$. But if $\lambda = 0$ at the optimal solution, then constraint (13) is not binding, which implies that χ^*, π^* must solve the following problem:

$$\max_{\{\chi,\pi\}} \pi\left[\int_{\chi}^{\overline{\theta}_G} \theta_G dH_G(\theta_G) - K\right]$$
$$\text{s.t.} \quad 0 \leq \pi \leq 1.$$

If $\pi^* > 0$ in the solution to the above problem, then the objective function is monotonically decreasing in χ over $\left[\max\{\underline{\theta}_G, 0\}, \overline{\theta}_G\right]$; thus it must be that $\chi^* = \max\{\underline{\theta}_G, 0\}$. Thus, if $\pi^* > 0$, it must maximize

$$\pi\left[\int_{\max\{\underline{\theta}_G,0\}}^{\overline{\theta}_G} \theta_G dH_G(\theta_G) - K\right] = \pi\left[\mathrm{E}(\theta_G|\theta_G \geq 0) - K\right].$$

By assumption $\mathrm{E}(\theta_G|\theta_G \geq 0) > K$, so the solution must be $\pi^* = 1$, and thus $\chi^* = \max\{\underline{\theta}_G, 0\}$. But, at $p^* = c, \chi^* = \max\{\underline{\theta}_G, 0\}$ and $\pi^* = 1$, the right hand side of the budget constraint (13) reads

$$\pi^*\left[\chi^*(1 - H_G(\chi^*)) - K\right] + (p^* - c)\left[1 - H_P(p^*)\right] = \max\{\underline{\theta}_G, 0\} - K = \max\{\underline{\theta}_G - K, -K\},$$

which is strictly negative, thus the budget constraint (13) is violated, a contradiction. Hence, $p^* > c$ in any solution to (12).

(Part 2): As Part 1 establishes that $p^* > c$ in any optimum, it follows that there is a strict budget surplus if $\pi^* = 0$ (the tax collected from the private goods due to $p^* > c$ is unspent). Instead, consider a positive public good provision probability π' as given by

$$\pi' = \frac{(p^* - c)\left[1 - H_P(p^*)\right]}{K} > 0.$$

By construction, constraint (13) is satisfied by the alternative simple mechanism $(\pi', \chi = 0, p^*)$. Clearly $(\pi', \chi = 0, p^*)$ improves the objective of (12) upon $(\pi^* = 0, \chi = 0, , p^*)$. A contradiction.

(Part 3): This is obvious if $\underline{\theta}_G > 0$, since $\chi^* = \pi^*\underline{\theta}_G$ would be non-distortionary. Therefore, suppose $\chi^* = 0$ and $\underline{\theta}_G \leq 0$. From condition (16), it follows that $\lambda\left[1 - H_G(0)\right] = 0$, which can only hold if $\lambda = 0$. But, from the proof of Part 1, if $\lambda = 0$, then $p^* = c$, which contradicts our conclusion in Part 1. ■

To summarize, under the standard regularity condition on the virtual valuation $x_J(\cdot)$, the best separate provision mechanism can be characterized by two prices, one for the public and one for the private good, and a provision probability for the public good. In particular, the sole government intervention in the private good market under the best separate provision mechanism can be interpreted as a "unit sales tax" in the amount of $p^* - c$. Otherwise, the operation of the private good market can be left completely to the private sector. In other words, in the best separate provision mechanism, the private good can be provided in a completely decentralized manner via a competitive market (subject to a sales tax). The only connection between the private and public goods is that the revenue from the unit tax on the private good is used as a cross-subsidy to partially fund the cost of the public good. This is not surprising, as the welfare loss from a small enough tax on the private good is of second order, whereas increasing the probability of public good provision or reducing access fees for the public good leads to a first-order welfare gain since we have under-provision of the public good at the constrained optimum. The intuition for the strictly positive public good user fee f^* is similar: excluding consumers with valuations just above zero leads to only a second order welfare loss, whereas the associated revenue from charging a positive f^* generates a first order welfare gain.

4 Public Provision of Both Goods

In this section we extend the policy instruments for the government such that it is able to condition the provision probability and price for each of the two goods on the reported valuations of *both goods*. We interpret this as *public provision of both goods* since such joint provision mechanisms would not have been implementable if the private good were traded anonymously in the private sector.

For tractability we will study small perturbations of the best separate provision mechanism characterized in Proposition 1. Specifically, we will simply add a price τ, which is the fee charged to a consumer who consumes the bundle consisting of both the public and the private goods. Hence, we study mechanisms of the form (π, f, p, τ) instead of the mechanisms of the form (π, f, p) characterized in Proposition 1. Now, f is the user fee for the consumption of the public good only and p is the price for the private good charged to those who do not get access to the public good. If $\tau \neq f + p$, it requires that the government be actively involved in provision of the private good because such a scheme is feasible for the government only if it could monitor the consumers' purchases of the private goods. We will show that, generally, charging $\tau \neq f + p$ improves welfare.

Our local argument is silent on what the optimal joint provision mechanism is. However, we are mainly interested in the *qualitative* question of whether public provision of a private good can be efficiency enhancing. As we show below that the perturbation outperforms the best separate provision mechanism we identified in Section 3, the constrained optimal joint provision mechanism must be one in which the government takes an active part in the provision of the private good.

One natural question is, does our analysis suggest that all private goods should be publicly

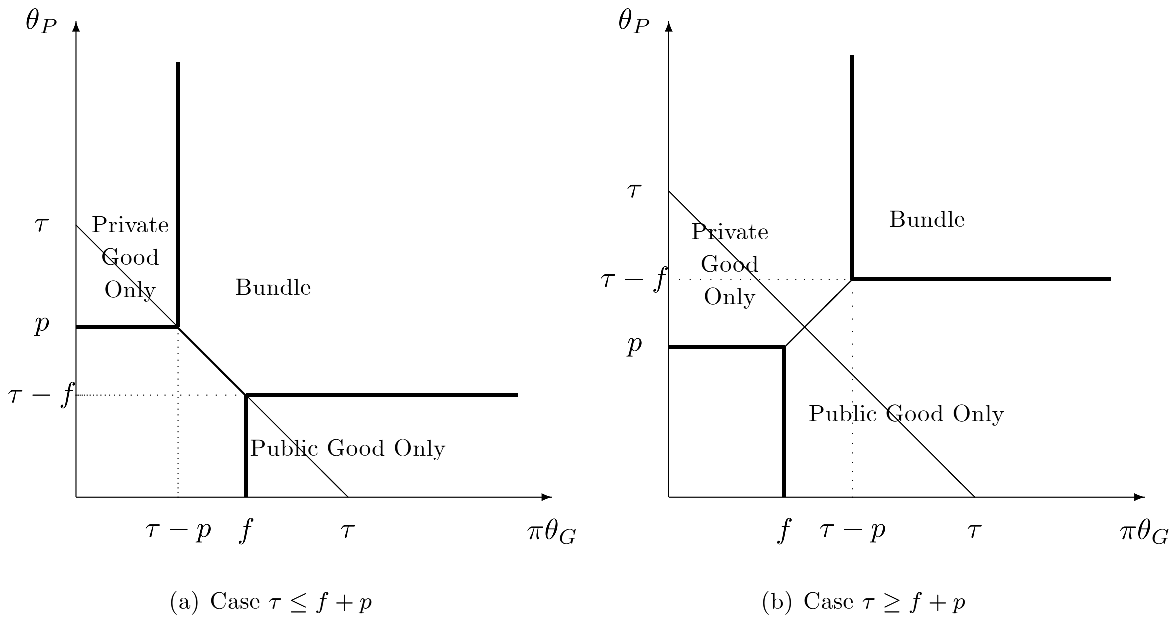

Figure 1: Consumer's Demand for Private Good Only, Public Good Only and the Bundle.

provided in an optimal mechanism? *The answer is no.* A mechanism of the form (π, f, p, τ) with the feature $\tau \neq f + p$ could be implemented only if consumers *cannot engage in arbitrage.* Thus, our analysis suggests that only private goods for which consumers cannot engage in price arbitrage could be used by the government to improve efficiency. Indeed, this no-arbitrage restriction seems realistic for many goods that are publicly provided in the real world. For example, publicly-provided education (including public colleges), public health insurance, and public healthcare are all commodities that are difficult or impossible to resell, thus preventing arbitrage. The necessity to rule out consumer arbitrage strengthens our argument, as it provides an explanation for why some, but not all, private goods are publicly provided.

4.1 Demands, Budget Surplus and Social Surplus

As a first step, we need to calculate the social surplus as well as the budget surplus under simple pricing mechanisms of the form (π, f, p, τ). To do this we first derive the demands for the two goods and the bundle for a given allocation rule (π, f, p, τ).

Given (π, f, p, τ), a type-(θ_G, θ_P) consumer's demand is easy to characterize: (1) she will demand *only the public good* if $\pi\theta_G - f \geq 0$, $\pi\theta_G - f \geq \theta_P - p$, and $\pi\theta_G - f \geq \pi\theta_G + \theta_P - \tau$; (2) she will demand *only the private good* if $\theta_P - p \geq 0$, $\theta_P - p \geq \pi\theta_G - f$, and $\theta_P - p \geq \pi\theta_G + \theta_P - \tau$; (3) she will demand the bundle if $\pi\theta_G + \theta_P - \tau \geq 0$, $\pi\theta_G + \theta_P - \tau \geq \pi\theta_G - f$, and $\pi\theta_G + \theta_P - \tau \geq \theta_P - p$. In each of the above three cases, which of inequalities are relevant depends on whether the bundle is cheaper or more expensive than the components. The two subfigures in Figure 1 plot the consumer's demand for the private good and public goods as a function of her valuations (θ_G, θ_P)

for the case $\tau \leq f+p$ and the case $\tau > f+p$ respectively. Notice that we have scaled down the valuation for the public good by the provision probability π in the graphs. Calculation based on Figure 1 immediately yields that the relevant demands can be summarized as in Table 1.

For notational simplicity, let $z=(\pi,f,p,\tau)$. Define by $G_1(z)$ the *budget surplus* (if positive) or budget deficit (if negative) given prices $z=(\pi,f,p,\tau)$ for the case $\tau \leq f+p$. Using the demands delineated in Table 1, we can write this as:

$$\begin{aligned} G_1(z) &= f\left[\int_{\frac{f}{\pi}}^{\bar{\theta}_G}\int_{\underline{\theta}_P}^{\tau-f} h(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right] + (p-c)\left[\int_{p}^{\bar{\theta}_P}\int_{\underline{\theta}_G}^{\frac{\tau-p}{\pi}} h(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right] \\ &\quad + (\tau-c)\left[\int_{\frac{\tau-p}{\pi}}^{\frac{f}{\pi}}\int_{\tau-\pi\theta_G}^{\bar{\theta}_P} h(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \int_{\frac{f}{\pi}}^{\bar{\theta}_G}\int_{\tau-f}^{\bar{\theta}_P} h(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right] - K\pi, \end{aligned} \tag{20}$$

Symmetrically, we let $G_2(z)$ denote the *budget surplus/deficit* given prices $z=(\pi,f,p,\tau)$ for the case $\tau \geq f+p$:

$$\begin{aligned} G_2(z) &= f\left[\int_{\frac{f}{\pi}}^{\frac{\tau-p}{\pi}}\int_{\underline{\theta}_P}^{\pi\theta_G+p-f} h(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \int_{\frac{\tau-p}{\pi}}^{\bar{\theta}_G}\int_{\underline{\theta}_P}^{p} h(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right] \\ &\quad + (p-c)\left[\int_{p}^{\tau-f}\int_{\underline{\theta}_G}^{\frac{\theta_P+f-p}{\pi}} h(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \int_{\tau-f}^{\bar{\theta}_P}\int_{\underline{\theta}_G}^{\frac{\tau-p}{\pi}} h(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right] + (\tau-c)\left[\int_{\frac{\tau-p}{\pi}}^{\bar{\theta}_G}\int_{\tau-f}^{\bar{\theta}_P} h(\boldsymbol{\theta})\,d\boldsymbol{\theta}\right] - K\pi. \end{aligned} \tag{21}$$

Note that $G_1(z)=G_2(z)$ when $\tau=f+p$ which can be seen by substituting $\tau=f+p$ into (20) and (21).

Next, let $S_1(z)$ denote the *social surplus* given prices $z=(\pi,f,p,\tau)$ for the case $\tau \leq f+p$,

$$\begin{aligned} S_1(z) &= \int_{\frac{f}{\pi}}^{\bar{\theta}_G}\int_{\underline{\theta}_P}^{\tau-f} \pi\theta_G h(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \int_{p}^{\bar{\theta}_P}\int_{\underline{\theta}_G}^{\frac{\tau-p}{\pi}} (\theta_P-c)\,h(\boldsymbol{\theta})\,d\boldsymbol{\theta} \\ &\quad + \int_{\frac{\tau-p}{\pi}}^{\frac{f}{\pi}}\int_{\tau-\pi\theta_G}^{\bar{\theta}_P} (\pi\theta_G+\theta_P-c)\,h(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \int_{\frac{f}{\pi}}^{\bar{\theta}_G}\int_{\tau-f}^{\bar{\theta}_P} (\pi\theta_G+\theta_P-c)\,h(\boldsymbol{\theta})\,d\boldsymbol{\theta} - K\pi. \end{aligned} \tag{22}$$

Symmetrically, let $S_2(z)$ be the *social surplus* given prices $z=(\pi,f,p,\tau)$ for the case $\tau \geq f+p$:

$$\begin{aligned} S_2(z) &= \int_{\frac{f}{\pi}}^{\frac{\tau-p}{\pi}}\int_{\underline{\theta}_P}^{\pi\theta_G+p-f} \pi\theta_G h(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \int_{\frac{\tau-p}{\pi}}^{\bar{\theta}_G}\int_{\underline{\theta}_P}^{\tau-f} \pi\theta_G h(\boldsymbol{\theta})\,d\boldsymbol{\theta} \\ &\quad + \int_{p}^{\tau-f}\int_{\underline{\theta}_G}^{\frac{\theta_P+f-p}{\pi}} (\theta_P-c)\,h(\boldsymbol{\theta})\,d\boldsymbol{\theta} + \int_{\tau-f}^{\bar{\theta}_P}\int_{\underline{\theta}_G}^{\frac{\tau-p}{\pi}} (\theta_P-c)\,h(\boldsymbol{\theta})\,d\boldsymbol{\theta} \\ &\quad + \int_{\frac{\tau-p}{\pi}}^{\bar{\theta}_G}\int_{\tau-f}^{\bar{\theta}_P} (\pi\theta_G+\theta_P-c)\,h(\boldsymbol{\theta})\,d\boldsymbol{\theta} - K\pi. \end{aligned} \tag{23}$$

For the same reasons as that for the budget surplus, $S_1(z)=S_2(z)$ when $\tau=f+p$.

4.2 Two Auxiliary Problems

In order to provide conditions for when public provision of the private good outperforms the best decentralized outcome, we construct two *auxiliary* optimization problems. In words, the problems define the optimal price vectors (π,f,p,τ) under the restrictions that $\tau \leq f+p$ and $\tau \geq f+p$

	Case 1: $\tau \leq f+p$	Case 2: $\tau \geq f+p$
Public Good Only	$\int_{\frac{f}{\pi}}^{\overline{\theta}_G} \int_{\underline{\theta}_P}^{\tau-f} h(\boldsymbol{\theta})\, d\boldsymbol{\theta}$	$\int_{\frac{f}{\pi}}^{\frac{\tau-p}{\pi}} \int_{\underline{\theta}_P}^{\pi\theta_G+p-f} h(\boldsymbol{\theta})\, d\theta + \int_{\frac{\tau-p}{\pi}}^{\overline{\theta}_G} \int_{\underline{\theta}_P}^{\tau-f} h(\boldsymbol{\theta})\, d\boldsymbol{\theta}$
Private Good Only	$\int_{p}^{\overline{\theta}_P} \int_{\underline{\theta}_G}^{\frac{\tau-p}{\pi}} h(\boldsymbol{\theta})\, d\boldsymbol{\theta}$	$\int_{p}^{\tau-f} \int_{\underline{\theta}_G}^{\frac{\theta_P+f-p}{\pi}} h(\boldsymbol{\theta})\, d\boldsymbol{\theta} + \int_{\tau-f}^{\overline{\theta}_P} \int_{\underline{\theta}_G}^{\frac{\tau-p}{\pi}} h(\boldsymbol{\theta})\, d\boldsymbol{\theta}$
Bundle	$\int_{\frac{\tau-p}{\pi}}^{\frac{f}{\pi}} \int_{\tau-\pi\theta_G}^{\overline{\theta}_P} h(\boldsymbol{\theta})\, d\boldsymbol{\theta} + \int_{\frac{f}{\pi}}^{\overline{\theta}_G} \int_{\tau-f}^{\overline{\theta}_P} h(\boldsymbol{\theta})\, d\boldsymbol{\theta}$	$\int_{\frac{\tau-p}{\pi}}^{\overline{\theta}_G} \int_{\tau-f}^{\overline{\theta}_P} h(\boldsymbol{\theta})\, d\boldsymbol{\theta}$

Table 1: Summary of Demands.

respectively. Specifically, let (π^*, f^*, p^*) be the best separate provision mechanism characterized in Proposition 1; the two auxiliary problems are as follows. The first problem is:

$$\begin{aligned} &\max_{(f,p,\tau)} S_1(\pi^*, f, p, \tau) \\ \text{s.t.} \quad & G_1(\pi^*, f, p, \tau) \geq 0 \\ & f + p - \tau \geq 0. \end{aligned} \tag{24}$$

Problem (24) defines the best simple pricing policy in the form of (π, f, p, τ) under the restriction that $\pi = \pi^*$ and $\tau \leq f + p$. That is, the public good provision probability is fixed at the level as in the best separate provision mechanism characterized in Proposition 1, and the bundle is restricted to be no more expensive than separate purchase of its components. It is important to note that $(f^*, p^*, f^* + p^*)$ is a feasible solution for (24). Similarly, the second auxiliary problem is:

$$\begin{aligned} &\max_{(f,p,\tau)} S_2(\pi^*, f, p, \tau) \\ \text{s.t.} \quad & G_2(\pi^*, f, p, \tau) \geq 0 \\ & \tau - f - p \geq 0. \end{aligned} \tag{25}$$

Problem (25) gives the best simple pricing policy in the form of (π, f, p, τ) under the restriction that $\pi = \pi^*$ and $\tau \geq f + p$. That is, the public good provision probability is fixed at the level as in the best separate provision mechanism characterized in Proposition 1, and the bundle is restricted to be no cheaper than separate purchase of its components. Again, $(f^*, p^*, f^* + p^*)$ is a feasible also for (25).

The reason for constructing these auxiliary optimization problems is that a necessary condition for the best separate provision mechanism to be optimal when joint provision is feasible is that $(f^*, p^*, f^* + p^*)$ solves *both* (24) and (25). Our main result is a sufficient condition under which $(f^*, p^*, f^* + p^*)$ cannot solve both problems (24) and (25). Hence, under the identified sufficient condition there must exist a perturbation that improves welfare relative to that achieved by the best separate provision mechanism. Consequently, the constrained optimal joint provision mechanism must also feature public provision of the private good.

Differentiating (20) and evaluating at $z^* = (\pi^*, f^*, p^*, f^* + p^*)$, we find that:[10]

$$\begin{aligned} \frac{\partial G_1(z^*)}{\partial f} &= \int_{\frac{f^*}{\pi^*}}^{\overline{\theta}_G} \{H_P(p^*|\theta_G) + (p^* - c) h_P(p^*|\theta_G)\} h_G(\theta_G) d\theta_G \\ &\quad - \frac{f^*}{\pi^*} H_P\left(p^*|\frac{f^*}{\pi^*}\right) h_G\left(\frac{f^*}{\pi^*}\right) \end{aligned} \tag{26a}$$

$$\begin{aligned} \frac{\partial G_1(z^*)}{\partial p} &= \int_{\underline{\theta}_G}^{\frac{f^*}{\phi_G^*}} [\{(1 - H_P(p^*|\theta_G)) - (p^* - c) h_P(p^*|\theta_G)\}] h_G(\theta_G) d\theta_G \\ &\quad + \frac{f^*}{\phi_G^*}\left[1 - H_P\left(p^*|\frac{f^*}{\pi^*}\right)\right] h_G\left(\frac{f^*}{\pi^*}\right) \end{aligned} \tag{26b}$$

$$\begin{aligned} \frac{\partial G_1(z^*)}{\partial \tau} &= \int_{\frac{f^*}{\phi_G^*}}^{\overline{\theta}_G} \{(1 - H_P(p^*|\theta_G)) - (p^* - c) h_P(p^*|\theta_G)\} h_G(\theta_G) d\theta_G \\ &\quad - \frac{f^*}{\phi_G^*}\left[1 - H_P\left(p^*|\frac{f^*}{\pi^*}\right)\right] h_G\left(\frac{f^*}{\pi^*}\right) \end{aligned} \tag{26c}$$

These expressions inform us about the effect on the budget when one slightly perturbs the relevant prices f, p and τ from $z^* = (\pi^*, f^*, p^*, f^* + p^*)$.

Likewise, if we differentiate $S_1(\cdot)$ in (22) and evaluate at $z = z^*$, we obtain:

$$\frac{\partial S_1(z^*)}{\partial f} = \int_{\frac{f^*}{\pi^*}}^{\overline{\theta}_G} (p^* - c) h_P(p^*|\theta_G) h_G(\theta_G) d\theta_G - \frac{f^*}{\pi^*} H_P\left(p^*|\frac{f^*}{\pi^*}\right) h_G\left(\frac{f^*}{\pi^*}\right) \tag{27a}$$

$$\frac{\partial S_1(z^*)}{\partial p} = -\int_{\underline{\theta}_G}^{\frac{f^*}{\pi^*}} (p^* - c) h_P(p^*|\theta_G) h_G(\theta_G) d\theta_G + \frac{f^*}{\pi^*}\left[1 - H_P\left(p^*|\frac{f^*}{\pi^*}\right)\right] h_G\left(\frac{f^*}{\pi^*}\right) \tag{27b}$$

$$\frac{\partial S_1(z^*)}{\partial \tau} = -\int_{\frac{f^*}{\pi^*}}^{\overline{\theta}_G} (p^* - c) h_P(p^*|\theta_G) h_G(\theta_G) d\theta_G - \frac{f^*}{\pi^*}\left[1 - H_P\left(p^*|\frac{f^*}{\pi^*}\right)\right] h_G\left(\frac{f^*}{\pi^*}\right). \tag{27c}$$

We can also write out explicitly the gradients for S_2 and G_2, which we omit here.

Our first preliminary result is that, evaluated at $z^* = (\pi^*, f^*, p^*, f^* + p^*)$, the partial derivatives of G_1 and G_2 are the same, and the partial derivatives of S_1 and S_2 also coincide. Letting $DG_i(z)$ and $DS_i(z)$ denote the gradient vectors for $i = 1, 2$, we thus have that:

Lemma 3 *$DG_1(z^*) = DG_2(z^*)$ and $DS_1(z^*) = DS_2(z^*)$*

Lemma 3 follows from straightforward but tedious algebra. It is analogous to the well-known "smooth pasting" condition in optimal control problem with switching points (see, e.g. Dixit 1993). Here, the relevant budget surplus function switch from G_1 to G_2 at z^*, and we have the analog of "value matching" of G_1 and G_2 at z^*, i.e., $G_1(z^*) = G_2(z^*)$. Lemma 3 simply states that the two functions are smoothly pasted at the switch point z^*. The same is true for the social surplus functions S_1 and S_2 at the switch point z^*.

Now we establish a useful lemma:

Lemma 4 *Let λ^* be the multiplier on constraint (13) corresponding to the solution (π^*, f^*, p^*) of problem (12). Also, let λ_i be the multiplier on the resource constraint $G_i(f, p, \tau; \pi^*)$ for $i = 1, 2$ in problem (24) and (25). Then,*

[10] The details of the derivations for (26) and (27) are available in an appendix from the authors' website.

1. $\lambda_1 = \lambda^*$ *if* z^* *solves problem* (24);

2. $\lambda_2 = \lambda^*$ *if* z^* *solves problem* (25).

Proof. First consider (24). If z^* solves the problem, the Kuhn-Tucker necessary conditions for a solution must be fulfilled at z^*. Hence, there must exist $\lambda_1 > 0$ and $\mu_1 \geq 0$ such that

$$\frac{\partial S_1(z^*)}{\partial f} + \lambda_1 \frac{\partial G_1(z^*)}{\partial f} + \mu_1 = 0 \tag{28a}$$

$$\frac{\partial S_1(z^*)}{\partial p} + \lambda_1 \frac{\partial G_1(z^*)}{\partial p} + \mu_1 = 0 \tag{28b}$$

$$\frac{\partial S_1(z^*)}{\partial \tau} + \lambda_1 \frac{\partial G_1(z^*)}{\partial \tau} - \mu_1 = 0 \tag{28c}$$

$$\mu_1 (f + p - \tau) = 0, \quad \mu_1 \geq 0 \tag{28d}$$

Using the expressions for the partial derivatives in (26) and (27), it is easy to check that:

$$\frac{\partial S_1(z^*)}{\partial f} + \frac{\partial S_1(z^*)}{\partial \tau} = -\frac{f^*}{\phi_G^*} h_G\left(\frac{f^*}{\pi^*}\right) \tag{29a}$$

$$\begin{aligned} \frac{\partial G_1(z^*)}{\partial f} + \frac{\partial G_1(z^*)}{\partial \tau} &= \int_{\frac{f^*}{\pi^*}}^{\overline{\theta}_G} h_G(\theta_G)\, d\theta_G - \frac{f^*}{\phi_G^*} h_G\left(\frac{f^*}{\pi^*}\right) \\ &= \left[1 - H_G\left(\frac{f^*}{\pi^*}\right)\right] - \frac{f^*}{\phi_G^*} h_G\left(\frac{f^*}{\pi^*}\right). \end{aligned} \tag{29b}$$

Combining (28a) and (28c), and using (29), we have that

$$-\frac{f^*}{\phi_G^*} h_G\left(\frac{f^*}{\pi^*}\right) + \lambda_1 \left\{\left[1 - H_G\left(\frac{f^*}{\pi^*}\right)\right] - \frac{f^*}{\phi_G^*} h_G\left(\frac{f^*}{\pi^*}\right)\right\} = 0. \tag{30}$$

This condition is the same as (16), one of the first order conditions to problem (12) when the goods are sold separately. It follows that $\lambda_1 = \lambda^*$, since otherwise (30) will be violated. This proves the first part.

For the second part, we note that the Kuhn-Tucker conditions for Problem (25) are

$$\begin{aligned} \frac{\partial S_2(z^*)}{\partial f} + \lambda_2 \frac{\partial G_2(z^*)}{\partial f} - \mu_2 &= \frac{\partial S_1(z^*)}{\partial f} + \lambda_2 \frac{\partial G_1(z^*)}{\partial f} - \mu_2 = 0 \\ \frac{\partial S_2(z^*)}{\partial p} + \lambda_2 \frac{\partial G_2(z^*)}{\partial p} - \mu_2 &= \frac{\partial S_1(z^*)}{\partial p} + \lambda_2 \frac{\partial G_1(z^*)}{\partial p} - \mu_2 = 0 \\ \frac{\partial S_2(z^*)}{\partial \tau} + \lambda_2 \frac{\partial G_2(z^*)}{\partial \tau} + \mu_2 &= \frac{\partial S_1(z^*)}{\partial \tau} + \lambda_2 \frac{\partial G_1(z^*)}{\partial \tau} + \mu_2 = 0 \\ \mu_2 (f + p - \tau) &= 0, \mu_2 \geq 0 \end{aligned}$$

where the first equality in each line follows from Lemma 3. The same argument applies. ■

Together, Lemmas 3 and 4 make the Kuhn-Tucker conditions for problem (24) comparable with those for problem (25).

4.3 The Main Result

Now our main result about the sufficient condition under which public provision of private goods will improve welfare over the best separate provision mechanism follows:

Proposition 2 *Let λ^* be the multiplier on constraint (13) corresponding to the solution (π^*, f^*, p^*) of problem (12). Then, there exists a feasible simple pricing policy of the form (f, p, τ) that generates a higher social surplus than the best separate provision mechanism whenever*

$$DS_1(z^*) + \lambda^* DG_1(z^*) \neq 0. \tag{31}$$

Proof. Suppose to the contrary, $(f^*, p^*, f^* + p^*)$ solves both problems (24) and (25). Lemma 4 then implies that the multiplier in each problem must be given by λ^*. Thus if z^* is the best simple pricing policy for problem (24), then

$$\begin{aligned} \frac{\partial S_1(z^*)}{\partial f} + \lambda^* \frac{\partial G_1(z^*)}{\partial f} + \mu_1 &= 0 \\ \frac{\partial S_1(z^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*)}{\partial p} + \mu_1 &= 0 \\ \frac{\partial S_1(z^*)}{\partial \tau} + \lambda^* \frac{\partial G_1(z^*)}{\partial \tau} - \mu_1 &= 0 \\ \mu_1 (f + p - \tau) = 0, \qquad & \mu_1 \geq 0. \end{aligned} \tag{32}$$

Similarly if z^* is the best simple pricing policy for problem (25), then by using Lemma 3, we have

$$\begin{aligned} \frac{\partial S_1(z^*)}{\partial f} + \lambda^* \frac{\partial G_1(z^*)}{\partial f} - \mu_2 &= 0 \\ \frac{\partial S_1(z^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*)}{\partial p} - \mu_2 &= 0 \\ \frac{\partial S_1(z^*)}{\partial \tau} + \lambda^* \frac{\partial G_1(z^*)}{\partial \tau} + \mu_2 &= 0 \\ \mu_2 (f + p - \tau) = 0, \qquad & \mu_2 \geq 0 \end{aligned} \tag{33}$$

Assume that $\mu_1 > 0$. Then, (32) implies that $\frac{\partial S_1(z^*)}{\partial f} + \lambda^* \frac{\partial G_1(z^*)}{\partial f} < 0$, which makes it impossible to find $\mu_2 \geq 0$ such that (33) holds. Symmetrically, if $\mu_2 > 0$, then $\frac{\partial S_1(z^*)}{\partial f} + \lambda^* \frac{\partial G_1(z^*)}{\partial f} > 0$, which makes it impossible to find $\mu_1 \geq 0$ such that (32) holds. Since z^* must solve both (24) and (25) for there to be no improvement we conclude that $\mu_1 = \mu_2 = 0$, or else there is some z better than z^*. The claim follows. ■

Because the simple pricing joint provision mechanism is a subset of all feasible joint provision mechanisms, an immediate implication of Proposition 2 is that whenever condition (31) is satisfied, the optimal joint provision mechanism must generate higher social welfare than the optimal separate provision mechanism.

4.4 Independence

Now we use Proposition 2 above to examine the case where θ_G and θ_P are independent. In this case, there is indeed always an improvement over the best separate provision policy.

Proposition 3 *Suppose that θ_G and θ_P are independent. Then*

$$\frac{\partial S_1(z^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*)}{\partial p} > 0.$$

Proof. When $h_P(\theta_P|\theta_G) = h_P(\theta_P)$ for all θ_P, we have that

$$\begin{aligned}
\frac{\partial G_1(z^*)}{\partial p} &= [1 - H_P(p^*) - (p^* - c) h_P(p^*)] H_G\left(\frac{f^*}{\pi^*}\right) + \frac{f^*}{\pi^*}[1 - H_P(p^*)] h_G\left(\frac{f^*}{\pi^*}\right) \\
&= \frac{(p^* - c) h_P(p^*)}{\lambda^*} H_G\left(\frac{f^*}{\pi^*}\right) + \frac{f^*}{\pi^*}[1 - H_P(p^*)] h_G\left(\frac{f^*}{\pi^*}\right)
\end{aligned}$$

where the second equality uses (17), the first order condition for p^* in the separate provision case. Next,

$$\frac{\partial S_1(z^*)}{\partial p} = -(p^* - c) h_P(p^*) H_G\left(\frac{f^*}{\pi^*}\right) + \frac{f^*}{\pi^*}[1 - H_P(p^*)] h_G\left(\frac{f^*}{\pi^*}\right)$$

Hence,

$$\begin{aligned}
\frac{\partial S_1(z^*)}{\partial p} + \lambda^* \frac{\partial G_1(z^*)}{\partial p} &= -(p^* - c) h_P(p^*) H_G\left(\frac{f^*}{\pi^*}\right) + \frac{f^*}{\pi^*}[1 - H_P(p^*)] h_G\left(\frac{f^*}{\pi^*}\right) \\
&\quad + \lambda^* \left\{\frac{(p^* - c) h_P(p^*)}{\lambda^*} H_G\left(\frac{f^*}{\pi^*}\right) + \frac{f^*}{\phi_G^*}[1 - H_P(p^*)] h_G\left(\frac{f^*}{\pi^*}\right)\right\} \\
&= (1 + \lambda^*) \frac{f^*}{\pi^*}[1 - H_P(p^*)] h_G\left(\frac{f^*}{\pi^*}\right) > 0. \qquad \blacksquare
\end{aligned}$$

Proposition 3 establishes that when the valuations for the private and public goods are stochastically independent, there is a *welfare improving* role for the public provision of private goods over the best separate provision mechanism we characterized in Proposition 1. It also implies, by continuity, that there exists an *open set* of joint distribution functions for θ_G and θ_P for which public provision of private goods can be preferred from an efficiency viewpoint.[11]

5 Related Literature

Literature on the Public Provision of Private Goods. It is useful to divide the existing literature on public provision of private goods into three strands. The first strand, as represented by Fernandez and Rogerson (1995) and Epple and Romano (1996), is a positive theory driven by political economy considerations. Epple and Romano (1996) consider a model where the level of a private good (say, "health care") and whether private purchases are permitted to supplement the public provision are determined by majority voting. They show that, under standard assumptions on voter preferences, a regime with positive government provision and no restriction on private supplement is majority preferred to one of either only market provision or only government provision. The key intuition is that the median voter, who has income below the population mean, receives a positive net transfer if private goods are publicly provided. Fernandez and Rogerson (1995) use a similar model (with the crucial difference that the private good is only partially subsidized) to explain how public provision of private goods may occur even if the incidence favors wealthier households.

[11] We conjecture is that condition (31) is satisfied *generically* in the sense that it holds for almost all joint distribution functions.

The second strand shows that public provision of private goods may serve as a tool for income redistribution from the rich to the poor, but it assumes, instead of explains, why in-kind transfers are preferred to cash transfers. For example, Besley and Coate (1991) considers a model where households may opt out from public provision and purchase a higher quality version of the good if they are dissatisfied with the quality of the publicly-provided good. If the willingness to pay for quality is increasing in wealth, mainly rich households opt out from the publicly provided private goods, implying that the system of public provision of private goods can serve as a transfer towards poor individuals.[12]

The third strand, to which our paper is most closely related, takes a normative perspective and identifies circumstances under which providing a private good in-kind is a more efficient transfer instrument than cash transfers. Blackorby and Donaldson (1988) studied a setup where consumers have private information about their preferences and showed that in-kind transfers can be desirable due to its screening role, which allows better targeted transfers.[13] Coate (1995) analyzed an environment where the rich has altruistic preferences towards the poor and would like to insure the poor's income risks. If the poor are given cash, they may still opt not to purchase insurance to exploit the Samaritan's Dilemma. As a result, the rich may prefer to give the poor an in-kind transfer of insurance to overcome the difficulty of committing not to provide *ex post* assistance.[14] Garratt and Marshall (1994) considered the case of the public financing of college education. They argue that public finance of college education (a private good) provides gambles that families desire.

Our paper complements the above literature by analyzing a full mechanism design problem faced by a social planner with no desire to redistribute across individuals. Notably, we do not impose any exogenous restrictions on the set of feasible policy instruments other than the natural participation and incentive compatibility constraints by the individuals for our comparison benchmark of what can be achieved by the market without public provision of private goods. In contrast, Fernandez and Rogerson (1995), Epple and Romano (1996), Besley and Coate (1991), Garratt and Marshall (1994), and many others do rely on exogenously imposed restrictions on the available policy instruments. However, the participation constraints imposed in our analysis of the best separate provision mechanism does imply that our theory is suited mostly as a theory for the public provision of private goods by local governments (see footnote 5).

The key role played by the public provision of private goods in our paper also differs substantially from targeted transfers, commitment problems, or demand for gambles. In our paper, public provision of private goods supplies the government with information about individuals' private good purchases that facilitates more efficient revenue extraction that be used to fund the typically under-provided public goods.

Also importantly, our paper not only explains that public provision of some private goods may

[12] Other arguments based on a desire to redistribute income can be found in Blomquist and Christiansen (1995, 1999) and Cremer and Gahvarib (1997). Also, Gahvari and Mattos (2007) extends Besley and Coate (1991).

[13] See also Nichols and Zeckhauser (1982) for similar arguments.

[14] A similar model is considered by Bruce and Waldman (1991).

improve efficiency, but also highlights that only private goods for which consumers can not engage in price arbitrage could be used by the government to improve efficiency. None of the existing literature has any predictions about what private goods may be publicly provided.

How our Problem Differs from McAffe et al. (1989)? Our expression $\partial G_1(z^*)/\partial p$ in (26b) is identical to the condition in Proposition 1 of McAfee *et al.* (1989). Expression (26a) can also be written in that form by reversing the roles of θ_G and θ_P. This is not a coincidence. The derivatives in (26a), (26b), and (26c) are the effects on profits given a marginal increase in f, p and τ respectively. In the case of McAfee *et al.* (1989), going from (26b) to their main result is relatively straightforward since they demonstrated that *profits* may be increased relative to separate pricing, by adding the bundle. Since they study profit maximization their analogue of (f^*, p^*) are chosen to solve a monopolist profit maximization problem under separate pricing. Thus their p^*, for example, must satisfy the first order optimality condition $[1 - H_P(p^*)] - (p^* - c) h_P(p^*) = 0$. If θ_G and θ_P are independent, the first term in (26b) becomes:

$$\begin{aligned} & \int_{\underline{\theta}_G}^{\frac{f^*}{\pi^*}} \{[1 - H_P(p^*|\,\theta_G)] - (p^* - c) h_P(p^*|\,\theta_G)\} h_G(\theta_G) d\theta_G \qquad (34) \\ = & \int_{\underline{\theta}_G}^{\frac{f^*}{\pi^*}} \{[1 - H_P(p^*)] - (p^* - c) h_P(p^*)\} h_G(\theta_G), \end{aligned}$$

which is equal to zero from the optimality condition of p^* in their problem. Thus it follows immediately from (26b) that a small increase in the price of the private good (or a small decrease in the price from the bundle) would increase the profits in the case of independence. It can also be verified that a small increase in the price of the public good also increases profits if θ_G and θ_P are stochastically independent.

Our problem differs from McAfee *et al.* (1989) in two respects. First of all, our goal is to demonstrate that bundling can increase *social welfare* rather than profits. Secondly, because (f^*, p^*) in our problem, as characterized in Proposition 1, are *not chosen to maximize profits*, we cannot use the first order conditions from the optimal separable provision mechanism in the same way as McAfee *et al.* (1989).

One may also be tempted to explain the public provision of private goods as simply a way for the government to maximize revenue collections. In order for such an interpretation to be valid, however, one has to explicitly model how the government will be using the collected revenues. Our theory is relevants along as governments are aiming to maximize some measure of social welfare, not just revenues alone.

6 Conclusion

This paper shows that public provision of private goods by a local government may be justified on pure efficiency grounds, even if the local government does not seek to redistribute resources. We believe that this is important for two reasons. Firstly, while the literature has identified situations where in-kind redistribution leads to improvements in how well targeted the transfers are, cash transfers are still superior to in-kind redistribution in many cases since the latter misallocates resources. Secondly, there are large scale programs that publicly provide private goods for which the redistributional effects are neutral or regressive. For these reasons we believe that a theory that does not rely upon a desire to redistribute is needed.

The theory developed in this paper is based on asymmetric information and the premise that governments also provide non-rival goods. In such an environment we show that public provision of a private good generates information that facilitates more efficient revenue extraction, which helps overcome inefficiencies in public good provision. Our main result establishes that public provision of the private good improves economic efficiency under a condition that is always fulfilled under independence and satisfied for a large set of joint distributions.

The explanation of publicly provided private goods advanced in this paper differs substantially from explanations emphasized in the existing literature, which typically rely on preferences for redistribution. Unlike most of the literature, we are also able to avoid ad hoc restrictions on the set of policy instruments, such as ruling out cash transfers. In contrast, we take a purist mechanism design perspective, so that the benchmark outcome without public provision of the private good is constrained optimal subject to incentive feasibility.

One of the most important assumptions in our model is that consumers cannot arbitrage the goods. If goods can be bartered between consumers, public provision is a useless tool for the government. We think this is a strength of our model as because in-state college tuition, public health insurance, and public schools are all commodities that are costly and/or difficult to resell. Thus our paper also provides an explanation of why some, but not all, private goods are publicly provided.

Finally, our analysis also exemplifies a more general point about the optimal taxation literature. Our model combines optimal commodity taxation with a decision on how to provide an excludable public good. We found that the marginal price for access to the public good for consumers that purchase the private good should be different from those who do not. Hence, our model is a stylized example where provision and user fees for the public good, and taxes on the private good must be jointly determined in order to achieve economic efficiency. Our paper thus illustrates that the standard practice of separating the question of how a given budget should be spent from the question of how a given tax revenue should be raised generates efficiency losses.

References

[1] Al-Najjar, Nabil (2004). "Aggregation and the Law of Large Numbers in Large Economies," *Games and Economic Behavior,* Vol. 47, 1-35.

[2] Besley, Timothy and Stephen Coate (1991). "Public Provision of Private Goods and the Redistribution of Income." *American Economic Review*, Vol. 81, No. 4, 979-984.

[3] Blackorby, Charles and David Donaldson (1988). "Cash versus Kind, Self-selection, and Efficient Transfers." *American Economic Review*, Vol. 78, No. 4, 691-700.

[4] Blomquist, Sören and Vidar Christiansen (1995). "Public Provision of Private Goods as a Redistributive Device in an Optimum Income Tax Model." *Scandinavian Journal of Economics*, Vol. 97, No. 4, 547-67.

[5] Blomquist, Sören and Vidar Christiansen (1999). "The Political Economy of Publicly Provided Private goods." Journal of Public Economics, 73(1), 31-54

[6] Blomquist, Sören and Vidar Christiansen (2007). "Public Provision of Private Goods and Nondistortionary Marginal Tax Rates." *mimeo,* University of Uppsala.

[7] Bruce, Neil and Michael Waldman (1991). "Transfers In-Kind: Why They Can Be Efficient and Nonpaternalistic." *American Economic Review,* Vol 81, No 5, 1345–1351.

[8] Boadway, Robin, Maurice Marchand, and Motohiro Sato (1998). "Subsidies versus Public Provision of Private Goods as Instruments for Redistribution." *Scandinavian Journal of Economics,* Vol. 100, No. 3, 545-564.

[9] Borgers, Tilman and Peter Norman (2008). "A Note on Budget Balance under Interim Participation Constraints: The Case of Independent Types." forthcoming, *Economic Theory.*

[10] Coate, Stephen (1995). "Altruism, the Samaritan's Dilemma, and Government Transfer Policy." *American Economic Review*, Vol 85, No. 1, 46-57.

[11] Cremer, Helmuth and Firouz Gahvarib (1997). "In-Kind transfers, Self-Selection and Optimal Tax Policy." *European Economic Review*, Vol. 41, No. 1, 97-114.

[12] d'Aspremont, Claude and Louis-Andre' Gérard-Varet (1979). "Incentives and Incomplete Information", *Journal of Public Economics,* 11, 1979, 25-45.

[13] Dixit, Avinash (1993). *Art of Smooth Pasting.* (Fundamentals of Pure and Applied Economics), Routledge.

[14] Epple, Dennis and Richard E. Romano (1996). "Public Provision of Private Goods." *Journal of Political Economy*, Vol. 104, No. 1, 57-84.

[15] Fang, Hanming and Peter Norman (2009). "Optimal Provision of Multiple Excludable Public Goods." mimeo, University of Pennsylvania and UNC-Chapel Hill.

[16] Fernandez, Raquel and Richard Rogerson (1995). "On the Political Economy of Education Subsidies." *Review of Economic Studies*, Vol. 62, No. 2 ,249-262.

[17] Gahvari, Firouz and Enlinson Mattos (2007). "Conditional Cash Transfers, Public Provision of Private Goods, and Income Redistribution." *American Economic Review,* Vol. 97, No. 1, 491-502.

[18] Garratt, Rod and John M. Marshall (1994). "Public Finance of Private Goods: The Case of College Education." *Journal of Political Economy*, Vol. 102, No. 3, 566-582.

[19] Hellwig, Martin F. (2005). "A Utilitarian Approach to the Provision and Pricing of Excludable Public Goods." *Journal of Public Economics,* Vol. 89, No. 11-12, 1981-2003.

[20] Hellwig, Martin F (2003). "Public Good Provision with Many Participants." *Review of Economic Studies,* Vol. 70, No. 2, 589-614.

[21] Mailath, George and Andrew Postlewaite (1990). "Asymmetric Information Bargaining Problems with Many Agents." *Review of Economic Studies,* Vol. 57, No. 2, 351-368.

[22] Mas-Colell, Andreu, Michael Whinston and Jerry R. Green (1995). *Microeconomic Theory.* Oxford University Press.

[23] McAfee, R. Preston, John McMillan and Michael D. Whinston (1989). "Multiproduct Monopoly, Commodity Bundling, and Correlation of Values." *Quarterly Journal of Economics,* Vol. 104, No. 1, 371–84.

[24] Myerson, Roger (1981). "Optimal Auction Design." *Mathematics of Operations Research,* Vol. 6, No. 1, 58-73.

[25] Nichols, Albert L. and Richard J. Zeckhauser (1982). "Targeting Transfers Through Restrictions on Recipients." *American Economic Review Papers and Proceedings*, Vol. 72, 301-17.

[26] Norman, Peter (2004). "Efficient Mechanisms for Public Goods with Use Exclusion." *Review of Economic Studies*, Vol. 71, 1163-1188.

[27] Ramsey, Frank P. (1927). "A Contribution to the Theory of Taxation." *Economic Journal*, Vol. 37, No 145, 47-61.

[28] Riley, John and Richard Zeckhauser (1983). "Optimal Selling Strategies: When to Haggle, When to Hold Firm." *Quarterly Journal of Economics,* Vol. 98, No. 2, 267-289.

[29] Schmitz, P. W. (1997). "Monopolistic Provision of Excludable Public Goods Under Private Information." *Public Finance,* Vol. 52, No. 1, 89-101.

[30] Stokey, Nancy L. (1979). "Intertemporal Price Discrimination." *Quarterly Journal of Economics,* Vol. 93, No. 3, 355-371

A Details About Lemma 1.

Let $(\pi^*, \phi_G^*, t_G^*, \phi_P^*, t_P^*)$ solve the planning problem (3). Because, as argued in (4), the incentive compability constraints (3b) can be separated for the private good and the public good, we know that, fix (ϕ_P^*, t_P^*), (π^*, ϕ_G^*, t_G^*) must solve:

$$\max_{(\pi,\phi_G,t_G)} \int_{\Theta} \left[\phi_G(\theta_G)\theta_G - t_G(\theta_G)\right] dH_G(\theta_G) \tag{A1}$$

$$\text{s.t.} \quad 0 \le \phi_G(\theta_G)\theta_G - t_G(\theta_G) - \phi_G(\widehat{\theta}_G)\theta_G + t_G(\widehat{\theta}_G), \quad \forall \theta_G, \widehat{\theta}_G \in \Theta_G, \tag{A2}$$

$$0 \le \phi_G(\theta_G)\theta_G - t_G(\theta_G) + \phi_P^*(\theta_P)\theta_P - t_P^*(\theta_P), \quad \forall (\theta_G, \theta_P) \in \Theta_G \times \Theta_P,$$

$$0 \le \int_{\Theta_G} t_G(\theta_G)\, dH_G(\theta_G) - K\pi + \underbrace{\int_{\Theta_P} t_P^*(\theta_P)\, dH_P(\theta_P) - \int_{\Theta_P} c\phi_P^*(\theta_P)\, dH_P(\theta_P)}_{\text{constant}} \tag{A3}$$

$$0 \le \phi_G(\theta_G) \le \pi.$$

The optimization problem (A1) takes the private good allocation rule (ϕ_P^*, t_P^*) from an optimal mechanism as given and solves for an optimal allocation of the public good conditional on the transfer between markets (the constant in A3) and the reservation utilities implied by (ϕ_P^*, t_P^*).

Symmetrically, fix (π^*, ϕ_G^*, t_G^*), (ϕ_P^*, t_P^*) must solve:

$$\max_{(\phi_P,t_P)} \int_{\Theta_P} \left[\phi_P(\theta_P)\theta_P - t_P(\theta_P)\right] dH_P(\theta_P) \tag{A4}$$

$$\text{s.t.} \quad 0 \le \phi_P(\theta_P)\theta_P - t_P(\theta_P) - \phi_P(\widehat{\theta}_P)\theta_P + t_P(\widehat{\theta}_P) \quad \forall \theta_P, \widehat{\theta}_P \in \Theta_P \tag{A5}$$

$$0 \le \phi_G^*(\theta_G)\theta_G - t_G^*(\theta_G) + \phi_P(\theta_P)\theta_P - t_P(\theta_P) \quad \forall (\theta_G, \theta_P) \in \Theta_G \times \Theta_P$$

$$0 \le \underbrace{\int_{\Theta_G} t_G^*(\theta_G)\, dH_G(\theta_G) - K\pi^*}_{\text{constant}} + \int_{\Theta_P} t_P(\theta_P)\, dH_P(\theta_P) - \int_{\Theta_P} c\phi_P(\theta_P)\, dH_P(\theta_P).$$

That is, taking the public good provision as given, (ϕ_P^*, t_P^*) solves for the least distorted allocation of private goods conditional on the transfer to the other market and reservation utilities implied.

Define the "indirect utility functions"

$$U_J(\theta_J) \equiv \theta_J \phi_J(\theta_J) - t_J(\theta_J), \tag{A6}$$

for $J = G, P$. A routine argument based on reasoning akin to the envelope theorem (see, e.g. Myerson 1981 or Mas-Colell *et al.* 1995, page 888) can be used to establish:

Lemma A1 *Suppose that the marginal density $h_J(\theta_J)$ is strictly positive on its support $\Theta_J = [\underline{\theta_J}, \overline{\theta_J}]$. Then, (ϕ_J, t_J) satisfies the incentive compatibility constraints in (A2) and (A5) respectively if and only if ϕ_J is weakly increasing in θ_J and*

$$U_J(\theta_J) = U_J(\widehat{\theta}_J) + \int_{\widehat{\theta}_J}^{\theta_J} \phi_J(x)\, dx \quad \forall \theta_J, \widehat{\theta}_J \in \Theta_J.$$

Equally routine procedures using the characterization in Lemma A1 show that the aggregate transfer revenues from the public goods fees and the private goods fees respectively can be determined uniquely from the utility of the lowest type and the provision rules.

Lemma A2 *Suppose that* (π, ϕ_G, t_G) *and* (ϕ_P, t_P) *satisfy the incentive compatibility constraints in (A2) and (A5) receptively if and only* ϕ_J *is weakly increasing in* θ_J *and*

$$\int_{\Theta_J} t_J(\theta_J)\, dH_J(\theta_J) = \int_{\Theta_J} \phi_J(\theta_J)\, x_J(\theta_J)\, dH_J(\theta_J) - U_J(\underline{\theta_J}).$$

Furthermore, it is without loss of generality assume that the participation constraint of type $(\underline{\theta}_G, \underline{\theta}_P)$ binds, so that

$$\phi^*(\underline{\theta}_G)\,\underline{\theta}_G - t^*_G(\underline{\theta}_G) + \phi^*_P(\underline{\theta}_P)\,\underline{\theta}_P - t^*_P(\underline{\theta}_P) = 0,$$

at a solution to problem (3). Higher types can mimic $(\underline{\theta}_G, \underline{\theta}_P)$, so, just like in unidimensional problems, incentive compatibility automatically implies that the participation constraints hold for higher types, provided that it is satisfied for the lowest type.

Lemma A2 together with the binding participation constraint for $(\underline{\theta}_G, \underline{\theta}_P)$ suggests a reformulation of the public good problem (A1) as Problem (6). Specifically, Problem (6) is derived from problem (A1) as follows: (1) the transfers are eliminated from the objective function by substitution from the feasibility constraint (A3), which is assumed to bind (all constants have been eliminated); (2) All redundant participation constraints have been eliminated.

Problem (7) is derived from problem (A4) analogously.