

Optimal Provision of Multiple Excludable Public Goods*

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Abstract

This paper studies the optimal provision mechanism for multiple excludable public goods. For a class of problems with symmetric goods and binary valuations, the optimal mechanism involves bundling if a regularity condition, akin to a hazard rate condition, on the distribution of valuations is satisfied. Relative to separate provision mechanisms, the optimal bundling mechanism may increase the asymptotic provision probability of socially efficient public goods from zero to one and decreases the extent of use exclusions. In the case with two goods, the optimal solution replicates the separate provision outcome if the regularity condition is violated.

Keywords: Public Goods Provision; Bundling; Exclusion

JEL Classification Number: H41.

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1 Introduction

This paper studies the constrained efficient mechanism for multiple excludable public goods. By “excludable public goods,” we mean goods that allow the provider to exclude consumers from access, but, nevertheless, are fully non-rival in consumption. There are many real-world examples of excludable public goods: cable TV, electronic journals, computer software and digital music files are all almost perfect examples. Importantly, most of these goods are provided in bundles. For example, cable TV customers purchase most programming as components of a few big bundles; access to electronic libraries are usually provided through site licences that allow access to every issue of every journal in the library; and digital music files, computer software and other digital files are commonly sold in a bundled format.

Most of the economics literature focuses on the implications of bundling on the revenue of *producers*.¹ To the best of our knowledge, a normative benchmark addressing the pros and cons of bundling for *consumers* and *social welfare*, while still explicitly incorporating the non-rival nature of these goods, does not yet exist. Our paper aims to fill this gap and is a step toward a better understanding of the welfare consequences of bundling of non-rival goods.²

We consider a simple model with M excludable public goods, satisfying standard (but strong) separability assumptions on both the supply and demand side. On the demand side, a consumer is described by a vector of single good valuations, and her willingness to pay for a bundle is the sum of the valuations of the goods included in the bundle; on the supply side, the cost of providing any good is independent of which other goods are also provided. Under these separability assumptions, the first-best benchmark is to provide good j whenever the average valuation of the good exceeds the per capita cost of provision, and to exclude no consumer from usage. There is thus no role for either bundling or use exclusion if there is perfect information.

If, instead, preferences are private information, as we assume in this paper, then consumers must be given appropriate incentives to truthfully reveal their willingness to pay. Together with self-financing and participation constraints, it is then impossible to implement the non-bundling full information first-best solution. Then, as we show in this paper, bundling is often useful because it facilitates revenue extraction from the consumers and thus relaxes a binding resource constraint.

In Section 2 we consider a relatively general setup, and characterize the form of optimal provision mechanisms in symmetric environments in a way that considerably reduces the dimensionality of the problem. These results are then applied to a special case considered in the remainder of the paper, where valuations for each good takes only two values, and the costs and valuation distributions are symmetric for the M goods. In this simplified setting we obtain a full characterization of the constrained efficient mechanism.

The solution is rather striking. When there are a large number of agents, the optimal mechanism either provides all goods with probability close to one, or provides all goods with probability close to zero. Which of these two scenarios applies depends on whether a *monopolist profit maximizer* that provides the goods for sure could break even. If a regularity condition on the valuation distribution – which can be

¹See, e.g., Adams and Yellen (1976), McAfee, McMillan and Whinston (1989), Armstrong (1996, 1999), Bakos and Brynjolfsson (1999), Fang and Norman (2006b) and Vincent and Manelli (2006).

²For the cable TV industry, see Crawford (2008), Cullen and Crawford (2007), Crawford and Yorokoglu (2009) and Yorokoglu (2009) for some recent studies on the welfare effects of *à la carte* pricing on the consumers. These studies assume that the provision of the programs will not be affected by the *à la carte* pricing regulation. See Section 5 for more discussions on the implications of our results on this issue.

interpreted much like a hazard rate condition – is satisfied, then the optimal mechanism also prescribes a simple rule for user access to the public goods that are provided. All agents will fall into one of three groups depending on their numbers of high valuation goods. A first group consists of all agents whose numbers of high-valuation goods strictly exceed a threshold. These agents are given access to the grand bundle consisting of all goods. A second group consists of all agents whose numbers of high-valuation goods are strictly lower than the threshold. For these agents access is granted only to the goods for which they have high valuations. Finally, there is a third group, consisting of those with exactly the threshold-level number of high valuation goods. These agents receive random access to their low valuation goods and full access to their high valuation goods.³

If the regularity condition discussed above is violated, it is in general difficult to identify the binding constraints. However, the case with two goods is both tractable and instructive. Violations of the regularity condition can then be interpreted as when the valuations of the two goods are “too positively correlated.” This invalidates the approach we use for solving the regular case. When the valuations of the two goods are “too positively correlated,” there are too few “mixed types” (those with one high and one low valuation) to justify giving them preferential treatment in terms of accessing their low-valuation good, leading to a violation of the monotonicity that is required for the standard approach. Surprisingly, the optimal solution for this case is in fact identical to the solution when both goods must be separately provided.

The tractability of the multidimensional mechanism design problem studied in this paper comes from the fact that types are naturally ordered in terms of the number of high valuation goods. This allows us to exploit some important similarities with unidimensional problems that help us determine which constraints are likely to bind. In addition, for unidimensional problems it is known that maximizing social surplus subject to budget and participation constraints leads to a Lagrangian characterization that can be interpreted as a compromise between profit and welfare maximization (see Hellwig 2003, and Norman 2004). In Hellwig (2003) and Norman (2004), incentive, resource and participation constraints can be combined into a single constraint that the monopolistic provider’s profit must be non-negative. This cannot be done in the multidimensional model considered in this paper. However, Lagrange multipliers of “adjacent constraints” are linked in a way that allows for an analogous characterization.

The remainder of the paper is structured as follows. Section 2 presents the model and some simplifying characterization results. Section 3 introduces the special case when valuations are binary and derives the optimal mechanism in the regular case. In Section 4 we use some special cases to better interpret the characterization in Section 3. In particular, we characterize the optimal mechanism when the regularity condition is violated in the two good case. Section 5 contains a brief discussion of the relevance of our analysis with respect to concrete anti-trust issues. Appendix A contains the proofs of some of the key results in Section 3.⁴

2 The Model

Section 2.1 lays out a fairly general model. Section 2.2 describes the set of randomized direct mechanisms in a somewhat nonstandard but useful way, and Section 2.3 sets up the mechanism design problem.

³Note, however, the third group of agents will be rather small when there are many goods.

⁴An online appendix available at the authors’ websites contains all the omitted proofs.

In Sections 2.4 and 2.5 we show that it is without loss of generality to consider a smaller and more tractable class of *simple, anonymous and symmetric mechanisms* under some additional symmetry restrictions on the environment. The main results of this section are Propositions 1 and 2, which are used to reduce the dimensionality of the design problem studied in the remainder of the paper.

2.1 The Environment

There are M *excludable* and indivisible public goods, labeled by $j \in \mathcal{J} = \{1, \dots, M\}$ and n agents, indexed by $i \in \mathcal{I} = \{1, \dots, n\}$. The cost of providing good j , denoted $C^j(n)$, is independent of which of the other goods are provided. Notice that $C^j(n)$ depends on n , the number of *agents* in the economy, and not on the number of *users*. This assumption captures the fully non-rival nature of the public goods. Indexing the cost by n also enables us to analyze large economies without making the public goods a “free lunch” in the limit. We therefore assume that $\lim_{n \rightarrow \infty} C^j(n)/n = c^j > 0$.⁵

Agent i is described by a valuation for each good $j \in \mathcal{J}$. Her type is given by a vector $\theta_i = (\theta_i^1, \dots, \theta_i^M) \in \Theta \subset \mathbb{R}^M$, and her preferences are represented by the utility function,

$$\sum_{j \in \mathcal{J}} \mathbb{I}_i^j \theta_i^j - t_i, \quad (1)$$

where \mathbb{I}_i^j is a dummy variable taking value 1 when i consumes good j and 0 otherwise, and t_i is the quantity of the numeraire good transferred from i to the mechanism designer. Preferences over lotteries are of the expected utility form.

The type θ_i is private information to agent i . Unlike much of the bundling literature, valuations for different goods for any agent are allowed to be correlated. However, types are independent *across agents*. We denote by F the joint cumulative distribution over θ_i . For brevity of notation, we denote a generic *type profile* by $\theta \equiv (\theta_1, \dots, \theta_n) \in \Theta^n$. In the usual fashion, we let $\theta_{-i} = (\theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_n)$.

2.2 Mechanisms

An outcome in our environment has three components: (1) which goods, if any, should be provided; (2) who are to be given access to the goods that are provided; and (3) how to share the costs. The set of feasible *pure* outcomes is thus

$$A = \underbrace{\{0, 1\}^M}_{\substack{\text{provision/no provision} \\ \text{for each good } j}} \times \underbrace{\{0, 1\}^{M \times n}}_{\substack{\text{inclusion/no inclusion} \\ \text{for each agent } i \text{ and good } j}} \times \underbrace{\mathbb{R}^n}_{\substack{\text{“taxes” for each} \\ \text{agent } i}}. \quad (2)$$

By the revelation principle, we only consider direct mechanisms for which truth-telling is a Bayesian Nash equilibrium. A pure direct mechanism maps Θ^n onto A , and a randomized mechanism is represented in analogy with the representation of mixed strategies in Aumann (1964). That is, let $\Xi \equiv [0, 1]$ and think of $\vartheta \in \Xi$ as the outcome of a fictitious lottery where, without loss of generality, ϑ is uniformly distributed and independent of θ . A *random direct mechanism* is then a measurable mapping $\mathcal{G} : \Theta^n \times \Xi \rightarrow A$. A conceptual advantage of this representation is that it allows for a useful decomposition in the sense that

⁵There is no need to give this assumption any economic interpretation. It is best viewed as a way to ensure that the public good provision problem remains “significant” when there are a large number of agents.

we now may write $\mathcal{G} = (\{\zeta^j\}_{j \in \mathcal{J}}, \{\omega^j\}_{j \in \mathcal{J}}, \tau)$ where

$$\begin{aligned} \text{Provision Rule:} \quad \zeta^j &: \Theta^n \times \Xi \rightarrow \{0, 1\} \\ \text{Inclusion Rule:} \quad \omega^j &: \Theta^n \times \Xi \rightarrow \{0, 1\}^n \\ \text{Cost-sharing Rule:} \quad \tau &: \Theta^n \rightarrow \mathbb{R}^n. \end{aligned} \tag{3}$$

We refer to ζ^j as the *provision rule* for good j , and interpret $E_{\Xi} \zeta^j(\theta, \vartheta)$ as the probability of provision for good j given announcements θ . The rule $\omega^j = (\omega_1^j, \dots, \omega_n^j)$ is the *inclusion rule* for good j , and $E_{\Xi} \omega_i^j(\theta, \vartheta)$ is interpreted as the probability that agent i gets access to good j when announcements are θ , conditional on good j being provided. Finally, $\tau = (\tau_1, \dots, \tau_n)$ is the *cost-sharing rule*, where $\tau_i(\theta)$ is the transfer from agent i to the mechanism designer given announced valuation profile θ . In principle, transfers could also be random, but the pure cost-sharing rule in (3) is without loss of generality due to risk neutrality.

2.3 The Design Problem

Let E_{-i} denote the expectation operator with respect to (θ_{-i}, ϑ) . A mechanism is *incentive compatible* if truth-telling is a Bayesian Nash equilibrium in the revelation game induced by \mathcal{G} , i.e., $\forall i \in \mathcal{I}, \theta \in \Theta^n, \hat{\theta}_i \in \Theta$,

$$E_{-i} \left[\sum_{j \in \mathcal{J}} \zeta^j(\theta, \vartheta) \omega_i^j(\theta, \vartheta) \theta_i^j - \tau_i(\theta) \right] \geq E_{-i} \left[\sum_{j \in \mathcal{J}} \zeta^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \omega_i^j(\hat{\theta}_i, \theta_{-i}, \vartheta) \theta_i^j - \tau_i(\hat{\theta}_i, \theta_{-i}) \right]. \tag{IC}$$

We also require that the project be self-financing. For simplicity, this is imposed as an *ex ante balanced-budget constraint*.⁶

$$E \left(\sum_{i \in \mathcal{I}} \tau_i(\theta) - \sum_{j \in \mathcal{J}} \zeta^j(\theta, \vartheta) C^j(n) \right) \geq 0. \tag{BB}$$

Non-rival nature of the public goods is built into (BB) because the cost depends only on the provision probabilities $\zeta^j(\theta, \vartheta)$, and not on the inclusion probabilities $\omega_i^j(\theta, \vartheta)$. Finally, we require that a voluntary participation, or *individual rationality*, condition be satisfied. Individual rationality is imposed at the interim stage as:

$$E_{-i} \left[\sum_{j \in \mathcal{J}} \zeta^j(\theta, \vartheta) \omega_i^j(\theta, \vartheta) \theta_i^j - \tau_i(\theta) \right] \geq 0, \quad \forall i \in \mathcal{I}, \theta_i \in \Theta. \tag{IR}$$

A mechanism is *incentive feasible* if it satisfies (IC), (BB) and (IR). Because utility is transferable, constrained *ex ante* Pareto efficient allocations solve a utilitarian planning problem, where a fictitious social planner seeks to maximize social surplus subject to the constraints (IC), (BB) and (IR).⁷ Thus a mechanism is *constrained efficient* if it maximizes

$$\sum_{j \in \mathcal{J}} E \zeta^j(\theta, \vartheta) \left[\sum_{i \in \mathcal{I}} \omega_i^j(\theta, \vartheta) \theta_i^j - C^j(n) \right], \tag{4}$$

⁶As shown in Borgers and Norman (2008) it is without loss of generality to consider a resource constraint in *ex ante* form.

⁷The qualifier *ex ante* is crucial for the equivalence. See Ledyard and Palfrey (2007) for a characterization of *interim* efficiency.

over all incentive feasible mechanisms.⁸

Ex post efficiency requires that good j be provided if and only if $\sum_{i \in \mathcal{I}} \theta_i^j \geq C^j(n)$, and that all agents receive full access whenever a good is provided. Since the *ex post* efficient allocation requires that no agent be excluded from usage, it is implementable if and only if a *non-excludable* public good can be efficiently provided under (IC), (BB) and (IR). But we know from Mailath and Postlewaite (1990) that this is impossible except for trivial cases. Our setup is thus a second best problem.

2.4 Simple Anonymous Mechanisms

We now consider mechanisms that are less complex than general mechanisms formulated in (3).

Definition 1 *A mechanism is called a simple mechanism if it can be expressed as a $(2M + 1)$ -tuple $g = (\rho, \eta, t) \equiv (\{\rho^j\}_{j \in \mathcal{J}}, \{\eta^j\}_{j \in \mathcal{J}}, t)$ such that for each good $j \in \mathcal{J}$,*

$$\begin{aligned} \text{Provision Rule:} & \quad \rho^j : \Theta^n \rightarrow [0, 1] \\ \text{Inclusion Rule:} & \quad \eta^j : \Theta \rightarrow [0, 1] \\ \text{Cost-sharing Rule:} & \quad t : \Theta \rightarrow \mathbb{R}, \end{aligned} \tag{5}$$

where ρ^j is the provision rule for good j , η^j is the inclusion rule for good j , and t is the transfer rule, same for all agents.

Definition 2 *A simple mechanism is called anonymous if for every $j \in \mathcal{J}$, $\rho^j(\theta) = \rho^j(\theta')$ for every $(\theta, \theta') \in \Theta^n \times \Theta^n$ such that θ' can be obtained from θ by permuting the indices of the agents.*

Simple mechanisms of the form (5) are less complex than general direct revelation mechanisms of the form (3) in several ways. First, inclusion and transfer rules are the same for all agents. Second, conditional on θ , the provision probability $\rho^j(\theta)$ is independent of all other provision probabilities as well as all inclusion probabilities. Third, the inclusion and transfer rules for any agent i are independent of θ_{-i} . Finally, all agents are treated symmetrically in terms of the transfer and inclusion rules. However, simple mechanisms as defined in (5) do allow for asymmetric treatment of the agents in terms of the provision rule, which is why we introduce the notion of *anonymous* mechanisms in Definition 2.

Our first results show that focusing on simple anonymous mechanisms is without loss of generality:

Proposition 1 *For any incentive feasible mechanism \mathcal{G} of the form (3), there exists a simple anonymous incentive feasible mechanism g that generates the same social surplus.*

The rough idea for Proposition 1 is as follows. Risk neutral agents care only about the perceived probability of consuming each good and the expected transfer. Therefore, there is nothing to gain from conditioning transfers and inclusion probabilities on θ_{-i} , or by making inclusion and provision rules conditionally dependent. Mechanisms on form (5) are therefore sufficient. Moreover, given an incentive feasible mechanism, permuting the roles of the agents leaves the surplus unchanged and all constraints

⁸All these constraints are noncontroversial if the design problem is interpreted as a private bargaining agreement. If the goods are government provided, the participation constraints (IR) may seem questionable. One defense in this context is that the participation constraint is a reduced form of an environment where agents may “vote with their feet.” Another defense is to view this as a reduced form for inequality aversion of the planner (see Hellwig 2005).

satisfied. Thus an anonymous incentive feasible mechanism that generates the same surplus as the initial mechanism can be obtained by averaging over the $n!$ permuted mechanisms.⁹

2.5 Symmetric Treatment of the Goods

Our next result, on which we rely heavily in Sections 3 and 4, identifies sufficient conditions under which it is without loss of generality to treat goods symmetrically. Given valuation profile θ and a one-to-one permutation mapping of the goods $P : \mathcal{J} \rightarrow \mathcal{J}$ with inverse P^{-1} , let $\theta_i^P = (\theta_i^{P^{-1}(1)}, \theta_i^{P^{-1}(2)}, \dots, \theta_i^{P^{-1}(M)})$ denote the permutation of agent i 's type by changing the role of the goods in accordance to P . The valuation profile obtained when the role of the goods is changed in accordance to P for every $i \in \mathcal{I}$ is denoted by $\theta^P \equiv (\theta_1^P, \dots, \theta_n^P)$.

Definition 3 *Mechanism g is symmetric if for every θ and every permutation $P : \mathcal{J} \rightarrow \mathcal{J}$,*

1. $\rho^{P^{-1}(j)}(\theta^P) = \rho^j(\theta)$ for every $j \in \mathcal{J}$;
2. $\eta^{P^{-1}(j)}(\theta_i^P) = \eta^j(\theta_i)$ for every $j \in \mathcal{J}$;
3. $t(\theta_i^P) = t(\theta_i)$.

Note that in the definition of a symmetric mechanism, the *same permutation of goods is applied for all agents*.¹⁰

Clearly, the environment must be symmetric if there is to be any hope for symmetric solutions. First, provision costs need to be identical across goods, i.e. $C^j(n) = C(n)$ for all j . Second, some symmetry assumption on valuations is also necessary. Symmetry of valuations is imposed by requiring that $\theta_i = (\theta_i^1, \dots, \theta_i^M)$ is an *exchangeable* random vector, which means that $F(\theta_i) = F(\theta'_i)$ if θ'_i is a permutation of θ_i . Notice that this still allows for valuations to be correlated.

Under these restrictions it is without loss of generality to consider symmetric mechanisms:

Proposition 2 *Suppose that θ_i is an exchangeable random vector and $C^j(n) = C(n)$ for all $j \in \mathcal{J}$. Then, for any simple anonymous incentive feasible mechanism g , there exists a symmetric simple anonymous incentive feasible mechanism that generates the same surplus as g .*

The idea is similar to that of Proposition 1, except that it is the roles of the goods that are permuted. For intuition, consider the case with two goods, and suppose that the two goods are treated asymmetrically. Reversing the role of the goods, an alternative mechanism that generates the same surplus is obtained. Averaging over the original and the reversed mechanism creates a symmetric mechanism where surplus is unchanged.¹¹ Incentive feasibility of the new mechanism follows from incentive feasibility of the original mechanism. The proof of Proposition 2 generalizes this procedure by permuting the goods ($M!$ possibilities) and creating a symmetric mechanism by averaging over these permuted mechanisms.

⁹The proof is more complex than simply randomizing with equal probabilities over the $n!$ permutations because inclusion and provision probabilities can be correlated.

¹⁰As an example, suppose that there are two agents and two goods, and that the valuation for each good is either h or l . In this case $\Theta = \{(h, h), (h, l), (l, h), (l, l)\}$. Consider the type profile $\theta = (\theta_1, \theta_2) = ((h, l), (l, h)) \in \Theta^2$. Applying the only non-identity permutation of the goods, i.e., $P(1) = 2$ and $P(2) = 1$, to all agents generates a type profile $\theta^P = (\theta_1^P, \theta_2^P) = ((l, h), (h, l))$. Definition 3 requires that the allocations for type profile $((l, h), (h, l))$ is the same as the allocation for $((h, l), (l, h))$ with goods relabeled and transfers unchanged.

¹¹Provision probabilities and taxes are given by straightforward averaging, but since inclusion and provision probabilities may be correlated the procedure is somewhat more involved for the inclusion rules.

3 The Case with Binary Valuations

In the remainder of the paper we assume that $C^j(n) = cn$ for all $j \in \mathcal{J}$.¹² In addition, the valuation for each good j is now either high or low, so that $\theta_i^j \in \{l, h\}$ for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$. The probability of any type $\theta_i \in \Theta \equiv \{l, h\}^M$ is denoted $\beta_i(\theta_i)$.¹³ Valuations may be dependent across goods for an individual agent, but valuation profiles are independent across agents, which allows us to denote the probabilities of type profiles $\theta \in \Theta^n$ and $\theta_{-i} \in \Theta_{-i} \equiv \Theta^{n-1}$ by $\beta(\theta) = \times_{i=1}^n \beta_i(\theta_i)$ and $\beta_{-i}(\theta_{-i}) = \times_{i' \neq i} \beta_{i'}(\theta_{i'})$ respectively.

For any $\theta_i \in \Theta$, write $m(\theta_i) \in \{0, \dots, M\}$ as the number of goods for which $\theta_i^j = h$, i.e.,

$$m(\theta_i) = \# \left\{ j \in \mathcal{J} : \theta_i^j = h \right\}. \quad (6)$$

Given any $m \in \{0, \dots, M\}$ there are $\frac{M!}{m!(M-m)!}$ types $\theta_i \in \Theta$ such that $\theta_i^j = h$ for exactly m goods. Since θ_i is an exchangeable random vector all these types are equally likely, the probability that an agent has a high valuation for exactly m goods, denoted by β_m , is given by

$$\beta_m = \frac{M!}{m!(M-m)!} \beta_i(\theta_i), \quad (7)$$

where θ_i is any type with m high valuations.¹⁴

Proposition 1 assures that it is without loss of generality to consider simple mechanisms in the sense of Definition 1. The design problem of Section 2.3 can thus be simplified as:¹⁵

$$\max_{\{\rho, \eta, t\}} \sum_{\theta \in \Theta^n} \beta(\theta) \sum_{j=1}^M \rho^j(\theta) \left[\sum_{i=1}^n \eta_i^j(\theta_i) \theta_i^j - cn \right] \quad (8)$$

$$\text{s.t. } 0 \leq \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta) \eta_i^j(\theta_i) \theta_i^j - t_i(\theta_i) - \left[\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta_{-i}, \theta_i') \eta_i^j(\theta_i') \theta_i'^j - t_i(\theta_i') \right] \\ \forall (\theta_i, \theta_i') \in \Theta^2 \text{ and } i \in \mathcal{I} \quad (9)$$

$$0 \leq \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta_{-i}, \mathbf{1}) \eta_i^j(\mathbf{1}) l - t_i(\mathbf{1}), \text{ where } \mathbf{1} \equiv (l, \dots, l), \forall i \in \mathcal{I} \quad (10)$$

$$0 \leq \sum_{i=1}^n \sum_{\theta_i \in \Theta} \beta_i(\theta_i) t_i(\theta_i) - \sum_{\theta \in \Theta^n} \beta(\theta) \sum_{j=1}^M \rho^j(\theta) cn \quad (11)$$

$$\rho^j(\theta) \in [0, 1] \quad \forall \theta \in \Theta^n \text{ and } j \in \mathcal{J} \quad (12)$$

$$\eta_i^j(\theta_i) \in [0, 1] \quad \forall \theta_i \in \Theta \text{ and } i \in \mathcal{I}, j \in \mathcal{J}. \quad (13)$$

In words, social surplus is maximized subject to the incentive compatibility constraints in (9), the participation constraint (10) for type-I, and the resource constraint (11). The constraints (12) and (13), finally, make sure that inclusion and provision rules always generate valid probabilities.

To simplify our discussions in the analysis below, we introduce the following terminology:

¹²Keeping the per capita costs constant simplifies notation, but is not necessary.

¹³ $\beta_i(\cdot)$ actually does not depend on i , the identity of the agent. For now, we index it by i for notational convenience.

¹⁴In the simplest case where valuations for different goods are i.i.d. with α being the probability that $\theta_i^j = h$, we have that for any θ_i such that $m(\theta_i) = m$, $\beta_i(\theta_i) = \alpha^m (1 - \alpha)^{M-m}$ and $\beta_m = \frac{M!}{m!(M-m)!} \alpha^m (1 - \alpha)^{M-m}$.

¹⁵The dimensionality of the problem could be reduced further by using Propositions 1 and 2, but it is notationally convenient to impose these additional symmetry restrictions at a later stage. As such, we are indexing η_i^j and t_i by i , even though we know from Propositions 1 and 2 that they do not need to depend on i in the optimal mechanism.

Definition 4 An incentive constraint (9) is referred to as:

1. a downward incentive constraint if $\theta_i^j \geq \theta_i'^j$ for all $j \in \mathcal{J}$;
2. an upward incentive constraint if $\theta_i^j \leq \theta_i'^j$ for all $j \in \mathcal{J}$;
3. a diagonal incentive constraint if there exists $j, k \in \mathcal{J}$ so that $\theta_i^j > \theta_i'^j$ and $\theta_i^k < \theta_i'^k$;
4. an adjacent incentive constraint if θ_i and θ_i' differ only in a single coordinate.

3.1 The Relaxed Optimization Problem

Based on intuition from unidimensional mechanism design problems (e.g. Matthews and Moore 1987) and similar to Armstrong and Rochet (1999), we now formulate a *relaxed problem* where all incentive constraints in (9) except the *downward adjacent* incentive constraints are removed.¹⁶ The relaxed problem is:

$$\max_{\{\rho, \eta, t\}} \sum_{\theta \in \Theta^n} \beta(\theta) \sum_{j=1}^M \rho^j(\theta) \left[\sum_{i=1}^n \eta_i^j(\theta_i) \theta_i^j - cn \right] \quad (14)$$

$$\text{s.t. } 0 \leq \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta) \eta_i^j(\theta_i) \theta_i^j - t_i(\theta_i) - \left[\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta_{-i}, \theta_i | l_k) \eta_i^j(\theta_i | l_k) \theta_i^j - t_i(\theta_i | l_k) \right] \quad (15)$$

$\forall \theta_i \in \Theta$, and $\forall k$ such that $\theta_i^k = h$, and $\forall i \in \mathcal{I}$
(10), (11), (12) and (13),

where we denote by $\theta_i | l_k$ the type that is obtained from θ_i if the k -th coordinate is changed from h to l . Recall from (6) that $m(\theta_i)$ is the number of goods with high valuations in θ_i , hence for every $i \in \mathcal{I}$ and every $\theta_i \in \Theta$ there are $m(\theta_i)$ downward adjacent incentives in (15).

As (14) is a linear program, existence of solutions is not completely obvious. However, a routine argument based on compactifying the constraint set and Weierstrass maximum theorem ensures that there are solutions to (14):

Lemma 1 *There exists at least one optimal solution to (14).*

In what follows we will first present a sequence of intermediate results that will be used to characterize the solution to the relaxed problem (14); then in Section 3.5 we will present conditions on the primitives to ensure that the solution to the relaxed problem also solves the full problem (8).

3.2 Relating the Multipliers

As we consider a symmetric environment, Proposition 2 applies; thus optimal provision and inclusion rules are symmetric. In addition, strong duality in linear programming ensures that the value of the multiplier associated with any of the downward adjacent incentive constraint (15) depends only on the number of goods for which the consumer has a high valuation. Formally,

¹⁶Also see Chapter 6 of Bolton and Dewatripont (2005) for a detailed analysis of multi-dimensional mechanism problem for a revenue-maximizing monopolist selling private goods.

Lemma 2 For every $m \in \{1, \dots, M\}$, it is without loss of generality to assume that there exists some $\lambda(m) \geq 0$ such that $\lambda(m)$ is the multiplier associated with every constraint (15) such that $m(\theta_i) = m$.

From now on, we therefore denote by $\lambda(m)$ the multiplier for all downward adjacent incentive constraints for types with $m \geq 1$ high valuations. Also, we let $\lambda(0)$ denote the multiplier associated with the participation constraint (10) for type $\mathbf{l} = (l, \dots, l)$, and Λ denote the multiplier to the resource constraint (11).

Our next result show that the multipliers $\lambda(m)$, $m \in \{0, \dots, M\}$ and Λ are systematically linked:

Lemma 3 For every $m \in \{0, \dots, M\}$, the value of $\lambda(m)$ is related to Λ in accordance with

$$\lambda(m) = \begin{cases} \Lambda & \text{if } m = 0, \\ \frac{(m-1)!(M-m)!}{M!} \Lambda \sum_{j=m}^M \beta_j = \frac{(m-1)!(M-m)!}{M!} \Lambda \Pr[m(\theta_i) \geq m] & \text{if } m = 1, \dots, M \end{cases} \quad (16)$$

where β_j is defined in (7).

Lemma 3 is a key step in solving (14). Its role is similar to the characterization of incentive feasibility in terms of a single integral constraint in unidimensional mechanism design problems (i.e., the approach in Myerson 1981 and others). In multidimensional problems, it is impossible to collapse all constraints into a single constraint. Instead, Lemma 3 allows us to *indirectly* relate all optimality conditions to a single constraint.

Lemma 3 can be interpreted as coming directly from duality in linear programming, but the easiest way to understand it is to consider the first order condition of problem (14) with respect to $t_i(\theta_i)$ where θ_i is a type with $m(\theta_i) = m \in \{1, \dots, M-1\}$ high valuation goods. The constraints in problem (14) where $t_i(\theta_i)$ appears can be delineated as follows. First, $t_i(\theta_i)$ appears in the incentive constraints (15). There are m ways to change a single h -coordinate into an l , so there are m different adjacent downward deviations from θ_i . In addition, there are $M-m$ types θ'_i with $m(\theta'_i) = m+1$ such that an adjacent downward deviation from θ'_i can “turn into” type θ_i . Second, $t_i(\theta_i)$ appears in the balanced budget constraint (11). With these observations, the optimality conditions to the program (14) with respect to $t_i(\theta_i)$ yields:

$$-\lambda(m)m + \lambda(m+1)(M-m) + \Lambda\beta_i(\theta_i) = 0. \quad (17)$$

When $m(\theta_i) = 0$, which is the case if and only if $\theta_i = \mathbf{l} = (l, \dots, l)$, there is no possible downward adjacent deviation from type- \mathbf{l} ; instead, $t_i(\mathbf{l})$ appears in the participation constraint (10). Thus the optimality condition with respect to $t_i(\mathbf{l})$ is:

$$-\lambda(0) + \lambda(1)M + \Lambda\beta_i(\mathbf{l}) = 0. \quad (18)$$

Similarly, when $m(\theta_i) = M$, which is the case if and only $\theta_i = \mathbf{h} = (h, \dots, h)$, there is no possible upward adjacent deviation to type- \mathbf{h} , thus the optimality condition with respect to $t_i(\mathbf{h})$ is:

$$-M\lambda(M) + \Lambda\beta_i(\mathbf{h}) = 0. \quad (19)$$

Because $\beta_M = \beta_i(\mathbf{h})$, it is immediate from (19) that the characterization in (16) holds for $m = M$. The rest of the proof is an induction argument using (17) and (18) recursively. The identity (7) is finally used to translate terms involving $\beta_i(\theta_i)$ into β_m , the probabilities of numbers of high-valuation goods.

A rough intuition for Lemma 3 is as follows. If an extra unit of revenue can be extracted from types with m high valuations without upsetting any constraint, then an extra unit can be extracted from all higher types as well. This follows as transfers enter linearly, so the payoff difference for an agent with m high valuations between truth-telling and announcing $m - 1$ high valuations depends only on the difference in the transfers. Multipliers are therefore proportional to the probability that the number of valuations exceeds m .

3.3 Optimal Inclusion Rules

Our next two lemmas establish some properties of the optimal inclusion rules when a public good is provided. Lemma 4 shows that individuals will always be able to access the goods for which they have high valuations. In contrast, Lemma 5 shows that access to goods for which the consumer has a low valuation is determined by the implications of allowing access on a non-trivial trade-off between social welfare and revenue extraction, where the weights on revenue extraction is a scaling of Λ , the multiplier Λ associated with the resource constraint (11).

Lemma 4 (Optimal Inclusion Rule for High Valuation Goods) *Suppose that $\theta_i \in \Theta$ such that $\theta_i^j = h$ and that $\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta_i, \theta_{-i}) > 0$. Then, $\eta_i^j(\theta_i) = 1$ in any optimal solution to (14).¹⁷*

Lemma 4 may appear to be the well-known “no distortion at the top” result from the unidimensional mechanism design problem; however, this is not an accurate interpretation as we are in a multi-dimensional setting where *a priori* it might be optimal to restrict access to high-valuation goods for types with too few high valuations. Instead, Lemma 4 states that an agent should be given access to her high-valuation goods *irrespective of* her total number of high-valuation goods. The result is indeed best understood in terms of the relationship between the multipliers in (17). Providing an extra high-valuation good to a type with m high valuations relaxes the downward adjacent incentive constraints (15) for every type with exactly m high valuations. There are m such incentive constraints, so the effect associated with higher utility from truth-telling for these types is $\lambda(m)mh$. However, giving access to a high-valuation good j makes it more tempting for a type with $m+1$ high valuations to announce only m high valuations. There are $M - m$ types that could change a single coordinate from high to low and pretend to have m high valuation goods. The negative effect on the utility of these types from mis-reporting as a type with m high valuations is $-\lambda(m+1)(M-m)h$. It then follows from (17) that the first positive effect from making truth-telling more appealing always dominates the second negative effect; hence all consumers always get access to their high-valuation goods.

The characterization of the optimal inclusion rule for low-valuation goods is more complicated. Some exclusions from access to low-valuation goods are essential in order to extract sufficient revenue to cover the costs; but at the same time, exclusions diminish social welfare. Optimal inclusions to low-valuation goods are therefore determined from a non-trivial trade-off between revenue extraction and social welfare maximization. It turns out that this trade-off is neatly captured by the functions $\{G_m(\Phi)\}_{m=0}^{M-1}$ given by

$$G_m(\Phi) = (1 - \Phi)(M - m)l\beta_m + \Phi \left[\beta_m(M - m)l - (h - l) \sum_{j=m+1}^M \beta_j \right], \quad (20)$$

¹⁷The condition $\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta_i, \theta_{-i}) > 0$ is needed in Lemmas 4 and 5 because the inclusion rules have no effect on either the objective function or the constraints when the conditional probability of provision is zero.

where $G_m(\Phi)$ is relevant for each type θ_i with $m(\theta_i) = m \in \{0, \dots, M-1\}$. We will first state the formal result, and then discuss why $G_m(\Phi)$ summarizes the compromise between welfare and profit maximization.

Lemma 5 (Optimal Inclusion Rule for Low Valuation Goods) *Suppose that $\theta_i \in \Theta$ with $m(\theta_i) = m \in \{0, \dots, M-1\}$ where $\theta_i^j = l$ for some $j \in \mathcal{J}$, and that $\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta_i, \theta_{-i}) > 0$. Then,*

$$\eta_i^j(\theta_i) = \eta(m) \equiv \begin{cases} 0 & \text{if } G_m(\Phi) < 0 \\ z \in [0, 1] & \text{if } G_m(\Phi) = 0 \\ 1 & \text{if } G_m(\Phi) > 0, \end{cases} \quad (21)$$

in any optimal solution to (14) where $\Phi = \frac{\Lambda}{1+\Lambda}$.

To interpret (20) and gain intuition for Lemma 5, imagine that all goods are provided with certainty and consider the following two potential inclusion rules: (1) All agents with at least m high valuations receive access to all the goods, while those with $m-1$ or fewer high valuations only receive access to their high valuation goods; (2) All agents with at least $m+1$ high valuations receive access to all the goods, while those with m or fewer high valuations only receive access to their high valuation goods. It is easy to see that $(M-m)l\beta_m$, the first term in (20), is the gain in *social welfare* if the planner changes the inclusion rule from (2) to (1). Less obvious is that the term in square bracket in (20) is the difference in *revenue* between the two inclusion rules for the provider of the public goods. To see this, note that under the first inclusion rule, the provider can charge $mh + (M-m)l$ to agents who consume all goods, and charge h per good to the remaining agents who only obtain access to their high valuation goods. The expected revenue under the first inclusion rule and the above pricing policy, denoted by $R(m)$, is then:¹⁸

$$R(m) = [mh + (M-m)l] \sum_{j=m}^M \beta_j + h \sum_{j=1}^{m-1} \beta_j j. \quad (22)$$

The expected revenue under the second inclusion rule and analogous pricing policy is $R(m+1)$. It follows from some algebra that:

$$R(m) - R(m+1) = \beta_m(M-m)l - (h-l) \sum_{j=m+1}^M \beta_j. \quad (23)$$

Thus, the term in square bracket in (20) represents the effect on the provider's *revenue*, which may be positive or negative, if the planner changes the inclusion rule from (2) to (1). Consequently, $G_m(\Phi)$ is simply a *weighted average of the effect on social welfare and revenue* from giving, instead of excluding, the agents with m high valuations access to all their low-valuation goods where Φ , the weight on revenue, is a normalization of the multiplier on the resource constraint (11).¹⁹

The optimal inclusion rule for low valuation goods as characterized in Lemma 5 thus states that if the trade-off is in favor of social welfare, i.e. when $G_m(\Phi) > 0$, then the agents with m high valuations should be provided with full access to all their low-valuation goods; if $G_m(\Phi) < 0$, then they should be excluded from their low-valuation goods.

¹⁸A profit maximizing monopolist would simply pick m to maximize $R(m)$ provided that the public good is provided.

¹⁹This property has been shown in uni-dimensional settings (see Hellwig 2003 and Norman 2004); but to the best of our knowledge, no analogous result multidimensional setting has been shown in the literature.

3.4 Optimal Provision Rules

We will now characterize the optimal provision rule. To simplify exposition, it is useful to introduce the following notation. For a given type profile $\theta = (\theta_1, \dots, \theta_n)$, we denote by $H^j(\theta, m)$ the number of agents who have a *high* valuation for good j and a total of m high valuations; similarly, we denote by $L^j(\theta, m)$ the number of agents with a *low* valuation for good j and a total of m high valuations.²⁰ The optimal provision rule can then be expressed in terms of $H^j(\theta, m)$, $L^j(\theta, m)$, and $G_m(\Phi)$ as follows:

Lemma 6 (Optimal Provision Rules) *The provision rule for good j in the optimal solution to (14) satisfies:*

$$\rho^j(\theta) = \begin{cases} 0 & \text{if } \sum_{m=1}^M H^j(\theta, m) h + \sum_{m=0}^M L^j(\theta, m) \frac{1}{\beta_m(M-m)} \max\{0, G_m(\Phi)\} - cn < 0 \\ z \in [0, 1] & \text{if } \sum_{m=1}^M H^j(\theta, m) h + \sum_{m=0}^M L^j(\theta, m) \frac{1}{\beta_m(M-m)} \max\{0, G_m(\Phi)\} - cn = 0 \\ 1 & \text{if } \sum_{m=1}^M H^j(\theta, m) h + \sum_{m=0}^M L^j(\theta, m) \frac{1}{\beta_m(M-m)} \max\{0, G_m(\Phi)\} - cn > 0. \end{cases} \quad (24)$$

Similar to the optimal inclusion rule (21), the optimal provision rule (24) can also be interpreted as a compromise between welfare and revenue maximization. To see this, recall that Lemma 4 established that all agents always get access to all high-valuation goods that are provided. Thus, the term $\sum_{m=1}^M H^j(\theta, m) h$ is the social surplus from all consumers with a high valuation for good j . To understand the second term in (24), recall that $G_m(\Phi)$ is the effect from giving an agent with m high valuations access to *all* their low-valuation goods. There are $L^j(\theta, m)$ agents with a low valuation for j and a total of m high valuations, but $L^j(\theta, m) G_m(\Phi)$ must be scaled by $\frac{1}{\beta_m(M-m)}$ in order to express the effect in the same units as $\sum_{m=1}^M H^j(\theta, m) h$. The factor $\frac{1}{M-m}$ is straightforward as $G_m(\Phi)$ measures the effect of providing access to *all* $M - m$ low-valuation goods to agents with m high valuations, so scaling by $\frac{1}{M-m}$ yields the relevant *per good* effect. The need to scale up by $\frac{1}{\beta_m}$ is a consequence of $G_m(\Phi)$ being expressed as an unconditional value, while (24) is expressed as a value conditional on a *fixed* type profile θ . Hence, the first two terms in (24) may be thought of as the effect from provision on a combination of social welfare and revenue, whereas the third term obviously is the associated cost of provision.

3.5 Linking the Relaxed Problem (14) and the Full Problem (8)

We are now in position to link the solution to the relaxed problem (14) with that of the full problem (8). Many steps of this analysis are similar to Matthews and Moore (1987), but the multidimensional nature of our environment leads to some important differences (see also Armstrong and Rochet 1996 for a similar analysis).

Analogous to the standard approach in solving unidimensional problems, a key condition for the solution to the relaxed problem (14) to also solve the full problem (8) is that the mechanism that solves the relaxed problem is *monotonic* in the following sense:

Definition 5 *A mechanism (ρ, η, t) is monotonic if $\eta_i^j(\theta_i) \leq \eta_i^j(\theta'_i)$ and $\rho^j(\theta_{-i}, \theta_i) \leq \rho^j(\theta_{-i}, \theta'_i)$ whenever $m(\theta_i) \leq m(\theta'_i)$ and $\theta_i^j \leq \theta'_i^j$.*

Lemma 7 below presents a sequence of intermediate results that if the solution to the relaxed problem (14) (ρ, η, t) is *monotonic* and all the downward adjacent constraints *bind*, then (ρ, η, t) also satisfies all the other incentive constraints in the full problem (8). Then Lemma 8 shows that the downward adjacent incentive constraints in (14) will indeed bind if (ρ, η, t) is monotonic and is *not ex post* efficient.

²⁰Note that $H^j(\theta, m)$ and $L^j(\theta, m)$ are simple accounting summaries.

Lemma 7 (Relations Among Various Incentive Constraints)

1. If the downward adjacent constraints in (14) **bind** at (ρ, η, t) , then all upward adjacent incentive constraints in (8) are satisfied.
2. If (ρ, η, t) is monotonic and that all downward (upward, respectively) adjacent incentive constraints hold, then all downward (upward, respectively) constraints in (8) are satisfied.
3. If (ρ, η, t) is monotonic and that all downward and upward incentive constraints hold, then all incentive constraints in (8) are satisfied.

Lemma 8 Let (ρ, η, t) be an optimal solution to (14). If (ρ, η, t) is **not** *ex post* efficient, then every downward adjacent incentive constraint and the participation constraint for type- $\mathbf{l} = (l, \dots, l)$ bind.

The basic idea behind Lemma 8 is simply that either some inclusion probabilities or provision probabilities may be increased if there is slack in some constraint. In either case, there is a welfare improvement. The proof, however, is complicated by the possibility that the inclusion and provision probability may be equal to one for the particular θ_i where there is slack in the constraint. To deal with this possibility we instead extract some additional revenue from type θ_i . An inductive argument shows that this additional revenue can be used to improve the allocation unless the solution is *ex post* efficient.

Proposition 3 then combines and summarizes the results in Lemmas 7 and 8:

Proposition 3 Let (ρ, η, t) be an optimal solution to (14). If (ρ, η, t) is monotonic and is **not** *ex post* efficient, then (ρ, η, t) is also an optimal solution to the full problem (8).

Now we provide a useful sufficient condition under which the solution to the relaxed problem (14) is indeed monotonic in the sense of Definition 5:

Proposition 4 Let (ρ, η, t) be a solution to the relaxed problem (14), then (ρ, η, t) is monotonic if $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ is strictly decreasing in m on $\{0, \dots, M-1\}$.

The sufficient condition identified in Proposition 4 is almost, but not quite, a hazard rate condition, as the term $\frac{1}{M-m}$ makes the condition different from a simple hazard rate condition. To understand the “almost hazard rate condition,” recall that the optimal inclusion rules for the relaxed problem (14), as characterized in Lemma 5, have the property that, for any agent with m high valuations, she either gets access to all goods or only her high-valuation goods. Allowing access to all goods is preferred in terms of social surplus, but may reduce the revenue. Specifically, with probability β_m an agent has exactly m high valuations, and such an agent is willing to pay an extra $(M-m)l$ for access to her low-valuation goods. On the other hand, with probability $\sum_{j=m+1}^M \beta_j$, an agent’s number of high-valuation goods exceeds m , and from such an agent the revenue is reduced by $(h-l)$. The “almost hazard rate condition” identified in Proposition 4 insures that $\frac{\sum_{j=m+1}^M \beta_j (h-l)}{\beta_m (M-m) l}$ decreases in m . This implies that if it is optimal to give agents with m high valuations access to their low-valuation goods, then it must also be optimal to give agents with more than m high valuations access to their low-valuation goods. That is, the condition makes sure that the optimal mechanism is monotonic.

In general, $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ may be non-monotonic, because $\beta_m (M-m)$ may locally decrease faster than $\sum_{j=m+1}^M \beta_j$. However, the important special case where valuations are independent satisfies the

“almost hazard rate condition,” because under independence, the tail of the distribution over m is thin so that $\sum_{j=m+1}^M \beta_j$ decreases rapidly as m increases:

Observation 1 *A sufficient condition for $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ to be strictly decreasing in m on $\{0, \dots, M-1\}$ is that the valuations for any goods j and $j' \neq j$ are independent.*

A useful corollary of Proposition 4 is that the inclusion rule for low valuation goods as characterized in Lemma 5 takes a sharp threshold rule if $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ is strictly decreasing in m :

Corollary 1 *Let (ρ, η, t) be a solution to (14). Suppose that $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ is strictly decreasing in m on $\{0, \dots, M-1\}$. Then, there exists some \tilde{m} such that:*

1. $\eta_i^j(\theta_i) = \eta(m) = 0$ for every θ_i with $\theta_i^j = l$ if $m(\theta_i) < \tilde{m}$;
2. $\eta_i^j(\theta_i) = \eta(m) = 1$ for every θ_i with $\theta_i^j = l$ if $m(\theta_i) > \tilde{m}$.

3.6 The Main Result

Now we provide our main asymptotic result regarding the limit of the sequences of exact optimal solutions to (14) as the number of agents n goes to infinity.

Because our asymptotic characterization is a limit of exact solutions to the finite problem, it is a *valid approximation* of the solution for a large, yet finite, economy. This is one of the key advantages of our approach over the approach of directly considering a mechanism design problem with a continuum of agents. Another problematic issue with mechanism design problem with a continuum of agents is one is forced to assume that aggregate variables are independent of individual announcements; as a result, incentive constraints are much harder, if possible at all, to formulate under the continuum approach. For example, a Groves-Clarke mechanism cannot even be formulated, despite the fact that such a mechanism is applicable and generates an efficient outcome for *any* finite economy.

Considering the asymptotic result as the number of the agents goes to infinity allows us to obtain a more easily interpretable characterization of the solutions when n is large. Note that our earlier characterizations of the optimal inclusion and provision rules for the finite case are still contingent on the multiplier on the resource constraint; the limiting characterization is easier to understand because it can be described without any reference to any endogenous multiplier. Conditions for the limiting case simplify for two reasons. First, terms such as $H^j(\theta, m)/n$ and $L^j(\theta, m)/n$ used in Lemma 6 to describe the optimal provision rule converge to their expectations by the Law of Large Number. The consequence is that provision probabilities for unusual realizations of θ can be largely ignored. Secondly, in a large economy, a version of the “Paradox of Voting” applies to the provision rules: an individual agent must have a negligible impact on provision decisions when n is large. This simplifies the analysis tremendously as the provision rule becomes almost constant in the relevant range as n is large enough.

Henceforth mechanisms are indexed by the number of agents n when needed. Using the central limit theorem one can establish that:

Lemma 9 *For each n , let (ρ_n, η_n, t_n) be a solution to problem (14). Then, $E[\rho_n^j(\theta) | \theta_i^j] - E[\rho_n^j(\theta) | \theta_i^j] \rightarrow 0$ as $n \rightarrow \infty$ for any j and any pair $\theta_i^j, \theta_i^j \in \Theta$.*

Lemma 9 immediately implies that $E[\rho_n^j(\theta) | \theta_i^j] - E[\rho_n^j(\theta)] \rightarrow 0$. Hence, all conditional probabilities appearing in the incentive constraints may be approximated by the *ex ante* probability of provision, a fact used extensively in the proof of Proposition 5.

Our main result below is about the provision probabilities and inclusion rules in the limit as n goes to infinity:

Proposition 5 *Suppose that $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ is strictly decreasing on $\{0, \dots, M-1\}$ and let $R(m)$ be the revenue from the inclusion and pricing rules as defined in (22). Let (ρ_n, η_n, t_n) be a solution to the full problem (8) when there are n agents in the economy. Then,*

1. **(Asymptotic Provision Probabilities)**

- (a) *If $\max_m R(m) - cM < 0$, then $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 0$ for all $j \in \mathcal{J}$;²¹*
- (b) *If $\max_m R(m) - cM > 0$, then $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 1$ for all $j \in \mathcal{J}$.*

2. **(Asymptotic Inclusion Rules Conditional on Provision)** *If $\max_m R(m) - cM > 0$, and m^* be the smallest m such that $R(m) - cM > 0$, then there exists $N < \infty$ such that if $n \geq N$,*

- (a) *for all type θ_i with $m(\theta_i) \geq m^*$, $\eta_n^j(\theta_i) = 1$ for all $j \in \mathcal{J}$;*
- (b) *for all type θ_i with $m(\theta_i) = m^* - 1$, $\eta_n^j(\theta_i) = 1$ for all j such that $\theta_i^j = h$, and*

$$\eta_n^j(\theta_i) \rightarrow \frac{R(m^*) - cM}{R(m^*) - R(m^* - 1)}$$

for all j such that $\theta_i^j = l$;

- (c) *for all type θ_i with $m(\theta_i) \leq m^* - 2$, $\eta_n^j(\theta_i) = 1$ for all j such that $\theta_i^j = h$ and $\eta_n^j(\theta_i) = 0$ for all j such that $\theta_i^j = l$.*

The proof of Proposition 5 requires quite a bit of technical work. Yet, the key idea is rather simple. Provided that the “almost hazard rate condition” is satisfied, Corollary 1 ensures that the optimal inclusion rules have a threshold characterization. The condition on whether $\max_m R(m) - cM$ is positive or negative is therefore a condition on whether it is at all possible to generate sufficient revenues to cover the costs if all the public goods are provided with certainty. Most work in the proof for Part 1 goes into establishing that a large economy is approximately an economy where the provision decisions are made *ex ante*, not conditioning on θ . The intuition is that when agents have a small influence on provisions, the welfare loss from making the provision decision *ex ante* is negligible.

Part 2 in Proposition 5 provides a simple limiting characterization of the optimal inclusion rule. As we already know from the finite case, all agents get access to all high valuation goods. Access to low valuation goods, however, is limited to agent with at least m^* high valuation goods, where m^* is determined as the smallest threshold for which the provider can make a positive profit if providing the good for sure. The reason for the mixing for types with $m^* - 1$ valuations is that if access is granted only to those with m^* high valuations and above there will be a strict budget surplus when n is large. There is therefore room to give partial access to types with $m^* - 1$ high valuations, which increases social surplus.

²¹More precisely, for this case $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 0$ for all $j \in \mathcal{J}$ holds for any sequence of *feasible* solutions to the full problem (8).

4 Some Special Cases

In this section, we describe some special cases. We first consider the case with a single good; We then study the case with two goods, where we are able to provide a complete characterization also for the case when the “almost hazard rate” condition in Proposition 5 is violated. We finally summarize our results in relation to the existing literature in Table 1.

4.1 A Single Public Good

Let α be the probability that an agent has a high valuation for the public good.²² With only a single good, the “almost hazard condition” in Proposition 5 is trivially satisfied, thus Proposition 5 is always applicable. Note from the definition of $R(m)$ in (22), we have:

$$R(m) = \begin{cases} l & \text{if } m = 0 \\ \alpha h & \text{if } m = 1. \end{cases}$$

Hence, Proposition 5 says that $\lim_{n \rightarrow \infty} E\rho_n(\theta) = 0$ if $\max\{\alpha h, l\} - c < 0$; and $\lim_{n \rightarrow \infty} E\rho_n(\theta) = 1$ if $\max\{\alpha h, l\} - c > 0$.

The case with $l \geq c$ is trivial: if the low valuation exceeds the cost of provision, it is first best efficient to always provide and never exclude any consumers from usage, which can be implemented by a uniform user fee of c .

When $l < c < \alpha h$, we know from Proposition 5 that the probability for access for type l goods converges as $n \rightarrow \infty$ to

$$\frac{R(1) - c}{R(1) - R(0)} = \frac{\alpha h - c}{\alpha h - l} \in (0, 1). \quad (25)$$

Moreover, $G_1(\Phi)$ as defined in (20) is equal to zero, thus the provision rule characterized in (24) simplifies to

$$\rho^j(\theta) = \begin{cases} 0 & \text{if } H^j(\theta, 1)h - cn < 0 \\ z \in [0, 1] & \text{if } H^j(\theta, 1)h - cn = 0 \\ 1 & \text{if } H^j(\theta, 1)h - cn > 0, \end{cases} \quad (26)$$

which, interestingly, is exactly the same provision rule that would be chosen by a profit-maximizing monopolist. Of course, a profit maximizer will not sell access to low valuation agents. Thus for the one good case, the welfare loss from a for-profit monopolist relative to the constrained social optimum is only due to over-exclusion, not due to under-provision, even for finite n . With more than one good, there is typically also under-provision by a for-profit monopolistic provider for finite n .

Separate Provisions of Many Excludable Public Goods. If there are $M > 1$ excludable public goods, but if the social planner is restricted to consider the provision of each of the public goods separately, then the optimal separate provision and exclusion rules for each public good is identical to the single good case described above, with $\alpha = \sum_{m=1}^M \frac{m}{M} \beta_m$ being the (marginal) probability that an agent has a high valuation for any particular good.²³

²²Note that when $M = 1$, $\alpha = \beta_1$ in our earlier notation. We introduce the new notation α so that our discussion for the one good case can be exactly mapped to the case of separation provision of multiple public goods we will discuss below.

²³Of course, this statement is true only under our maintained assumption that there are no complementarities in preferences and production costs.

It is also clear that the results for the one good case applies to the case with multiple goods when valuations for different goods are perfectly correlated.

4.2 Two Goods

Here we describe the results for the two-good case.²⁴ For the two goods case, we can also characterize the optimal mechanism when the “almost hazard rate” regularity condition is violated. Most of the existing literature on multiple product mechanism design focused on two goods cases, e.g. Armstrong (1996) and Armstrong and Rochet (1999). It turns out that when there are only two goods, the “almost hazard rate” regularity condition is reduced to a “not too positively correlated” condition (e.g. Armstrong and Rochet 1999).

4.2.1 When the “Almost Hazard Rate” Condition is Satisfied

It is easily verified that, with two goods, $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ is strictly decreasing in m on $\{0, 1\}$ if and only if

$$\frac{\beta_1}{2} > \frac{\beta_0 \beta_2}{1 - \beta_0}, \quad (27)$$

which can be interpreted as saying that the valuation of the two public goods are “not too positively correlated.” Furthermore, it can be verified that

$$R(m) = \begin{cases} 2l & \text{for } m = 0 \\ (\beta_1 + \beta_2)(h + l) & \text{for } m = 1 \\ 2\alpha h & \text{for } m = 2, \end{cases}$$

where $\alpha = \frac{\beta_1}{2} + \beta_2$ is the marginal probability that an agent has a high valuation for any good. Ruling out the trivial case of $l \geq c$, we see from Proposition 5 that there are three possibilities:

- If $\max\{(\beta_1 + \beta_2)(h + l), 2\alpha h\} < 2c$, then $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 0$ for $j \in \{1, 2\}$;
- If $(\beta_1 + \beta_2)(h + l) > 2c$, then $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 1$ for $j \in \{1, 2\}$; all agents with at least one high valuation good get access to both goods, and those with only low valuations get access to each good with probability

$$\frac{R(1) - 2c}{R(1) - R(0)} = \frac{(\beta_1 + \beta_2)(h + l) - 2c}{(\beta_1 + \beta_2)(h + l) - 2l} \in (0, 1);$$

- If $2\alpha h > 2c > (\beta_1 + \beta_2)(h + l)$, then $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 1$ for $j \in \{1, 2\}$; all agents get access to their high valuation goods, and those with two low valuations do not get any access at all, but those with one high valuation get access to their low valuation good with probability

$$\frac{R(2) - 2c}{R(2) - R(1)} = \frac{2\alpha h - 2c}{2\alpha h - (\beta_1 + \beta_2)(h + l)} \in (0, 1). \quad (28)$$

It is worth emphasizing that in the optimal *joint* provision mechanism, both goods are provided with probability one asymptotically given that $\max\{(\beta_1 + \beta_2)(h + l), 2\alpha h\} > 2c$. In contrast, in the best *separate* provision mechanism characterized in Section 4.1, the good is provided asymptotically only

²⁴In an earlier working paper, Fang and Norman (2006a), we exclusively focused on the two goods case.

if $\alpha h > c$. There is a non-empty parameter region where $(\beta_1 + \beta_2)(h + l) > 2c > 2\alpha h$ where we get asymptotic non-provision if goods are provided separately, but the optimal bundling mechanism provides both goods for sure.

The increased provision probability for efficient public goods under a bundling mechanism relative to the separate provision mechanism is only one channel through which bundling may increase efficiency. Additionally, the optimal bundling mechanism creates welfare gains by increasing the probability of inclusion for low-valuation agents, an effect that is present also if the goods can be provided without bundling. To see this, suppose that $\alpha h > c$ so that both public goods are asymptotically provided with probability one with or without bundling. From (25), we know that under the best separate provision mechanism, the probability for access to a low valuation agent is $(\alpha h - c) / (\alpha h - l)$. In contrast, (28) implies that the *ex ante* probability for access conditional on a low valuation for the case where $2c > (\beta_1 + \beta_2)(h + l)$ is

$$\underbrace{\frac{\beta_1/2}{\beta_0 + \beta_1/2}}_{\text{prob of mixed type given a low value}} \times \underbrace{\frac{2\alpha h - 2c}{2\alpha h - (\beta_1 + \beta_2)(h + l)}}_{\text{access prob to low valuation good for mixed type from (28)}}. \quad (29)$$

Some algebra shows that (29) is larger than $\frac{\alpha h - c}{\alpha h - l}$ whenever $\frac{\beta_1}{2} > \frac{\beta_0 \beta_2}{1 - \beta_0}$, which is precisely the condition under which Proposition 5 is applicable. Fewer consumers are thus excluded in the optimal bundling mechanism. A similar calculation applies to the case with $(\beta_1 + \beta_2)(h + l) > 2c$.

4.2.2 When the ‘‘Almost Hazard Rate’’ Condition is Violated

The case with two goods also provides a useful setup for investigating the case when the regularity condition on $\frac{1}{M-m} \frac{\sum_{j=m+1}^M \beta_j}{\beta_m}$ fails. As shown in (27) this reduces to the condition that $\frac{\beta_1}{2} \leq \frac{\beta_0 \beta_2}{1 - \beta_0}$, which may be interpreted as saying that the valuations are (sufficiently strongly) positively correlated. In the Appendix we prove that the asymptotic characterization for this case is:

Proposition 6 *Assume that $\frac{\beta_1}{2} \leq \frac{\beta_0 \beta_2}{1 - \beta_0}$ and $l < c$. Then:*

1. $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 0$ for every j if $\alpha h < c$ for any sequence $\{\rho_n, \eta_n, t_n\}$ of feasible mechanisms.
2. $\lim_{n \rightarrow \infty} E\rho_n^j(\theta) = 1$ for every j if $\alpha h > c$ for any sequence $\{\rho_n, \eta_n, t_n\}$ of optimal mechanisms. Moreover, all consumers get access to the high valuation goods and

$$\eta_n^j(\theta_i) \rightarrow \frac{\alpha h - c}{\alpha h - l} \in (0, 1)$$

as $n \rightarrow \infty$ for every θ_i with $\theta_i^j = l$.

Surprisingly, Proposition 6 shows that the solution is identical to the case when bundling is not allowed.²⁵ To understand why, recall that asymptotic provision or non-provision is related to whether the maximal revenue for a monopolistic provider of the goods – if provided – exceeds the costs. The revenue maximizing selling strategy for a monopolist, if both public goods are provided, is either to sell

²⁵This is *not* inconsistent with the results in McAfee, McMillan and Whinston (1989) and Jehiel, Meyer-ter-Vehn and Moldovanu (2007). They showed that for ‘‘generic’’ continuous valuation distributions, a monopolist seller’s revenue will be higher under mixed bundling in posted pricing and auction settings respectively. Our distributions are discrete.

goods separately at price h , or sell the goods as a bundle at price $h + l$, or to charge l for each good. These selling strategies generate a revenue of $2\alpha h$, $(\beta_1 + \beta_2)(h + l)$, and $2l$ respectively. Since we are already assuming $l < c$, the question is thus whether $\max\{(\beta_1 + \beta_2)(h + l), 2\alpha h\}$ exceeds $2c$. For the first part of Proposition 6, if $\alpha h < c$ and $l < c$ are both satisfied, we have that

$$\begin{aligned} (\beta_1 + \beta_2)(h + l) &< (\beta_1 + \beta_2)(h + c) < (\beta_1 + \beta_2) \left(\frac{1}{\alpha} + 1 \right) c \\ &= \left[2 + \frac{(1 - \beta_0) \left(\frac{1}{2}\beta_1 + \beta_2 \right) - \beta_2}{\frac{1}{2}\beta_1 + \beta_2} \right] c < 2c \end{aligned}$$

when $\frac{1}{2}\beta_1 \leq \frac{\beta_0\beta_2}{1-\beta_0}$. This calculation shows that it is impossible to provide the goods with probability 1 if $\alpha h < c$, and the idea for why this translates into $E\rho_n^j(\theta) \rightarrow 0$ is the same as for the previous analysis.

For the second part of Proposition 6, consider the case when either separate provision or bundling provide sufficient revenue to cover the cost of provision. In the solution characterized by Proposition 6, the asymptotic *ex ante* probability of getting access to low valuation good j is

$$\left(\beta_0 + \frac{1}{2}\beta_1 \right) \frac{\alpha h - c}{\alpha h - l}.$$

If, instead, the mechanism which is optimal for the case with $\frac{\beta_1}{2} > \frac{\beta_0\beta_2}{1-\beta_0}$ is used, the *ex ante* probability of getting access to low valuation good is

$$\beta_0 \frac{(\beta_1 + \beta_2)(h + l) - 2c}{(\beta_1 + \beta_2)(h + l) - 2l}.$$

Some algebra along the lines discussed in connection with (29) shows that the *ex ante* probability of getting access and therefore also the social surplus is actually smaller using the bundling mechanism in this case. Ultimately, this is driven by the relative scarcity of mixed type agents.

4.2.3 Relationship to the Existing Literature

Table 1 summarizes our results on the asymptotic provision probabilities under different bundling and exclusion scenarios for the two good case, and contrasts our results with those in the literature.²⁶

Bundling \ Exclusion	No Exclusion	Exclusion
No Bundling	$E\rho_n^{j*} \rightarrow 0$ (Mailath and Postlewaite 1990)	$E\rho_n^{j*} \rightarrow 0$, if $\alpha h < c$ $E\rho_n^{j*} \rightarrow 1$, if $\alpha h > c$ (Norman 2004)
Bundling Allowed	$E\rho_n^{j*} \rightarrow 0$ (Mailath and Postlewaite 1990)	Case 1: $\frac{\beta_1}{2} > \frac{\beta_0\beta_2}{1-\beta_0}$ Case 2: $\frac{\beta_1}{2} \leq \frac{\beta_0\beta_2}{1-\beta_0}$ $E\rho_n^{j*} \rightarrow 0$, if $\max_m R(m) < 2c$ $E\rho_n^{j*} \rightarrow 0$, if $\alpha h < c$ $E\rho_n^{j*} \rightarrow 1$, if $\max_m R(m) > 2c$ $E\rho_n^{j*} \rightarrow 1$, if $\alpha h > c$ (This Paper)

Table 1: The Asymptotic Provision Probability under Different Bundling and Exclusion Scenarios.

²⁶Mailath and Postlewaite (1990) considers a single-dimensional problem without use exclusion. However, the probabilities of provision in a multidimensional setting allowing for bundling can be bounded from above by a single-dimensional problem, where the valuation is the maximum of the individual good valuations.

5 Conclusion and Discussion

This paper studies the welfare maximizing provision mechanism for multiple excludable public goods when agents' valuations are private information. For a parametric class of problems with M goods whose valuations take binary values, we fully characterize the optimal mechanism and demonstrate that it involves bundling if a regularity condition, akin to a hazard rate condition, on the distribution of valuations is satisfied. Bundling improves the allocation in two ways. First, it may increase the asymptotic provision probability of socially efficient public goods from zero to one. Second, the extent of use exclusion is decreased when provision is bundled. For the case of two goods, we also show that if the regularity condition is violated, then the optimal solution replicates the separate provision outcome.

By studying the optimal, instead of revenue-maximizing, mechanism for the provision excludable public goods, our paper highlights the potential importance of bundling on *social welfare* for a variety of markets where the goods are non-rival but potentially excludable in use. Thus our analysis can be viewed as a *positive* theory of bundling in these markets. It can also be viewed as a step toward the development of a useful *normative* benchmark for bundling for markets of excludable public goods, or, more generally, goods with large fixed costs. Our model is highly stylized, it still has enough flexibility to generate a non-trivial trade-off that with potentially important anti-trust implications. Our analysis shows that it is sometimes possible that bundling is *required* for the monopolist provider of the public goods to break even. In such cases the profit-maximizing outcome with bundling is better for the consumer than the welfare maximizing outcome without bundling; thus a requirement to “unbundle” is strictly worse for the consumers because the public goods would simply not be provided when bundling is not allowed.²⁷

One of the best examples where our analysis may be relevant is the recent regulations considered by U.S. Congress and the Federal Communications Commission (FCC) requiring *à la carte* pricing of cable channels. The few existing economics studies, e.g., Crawford (2008), Cullen and Crawford (2007), Crawford and Yorokoglu (2009) and Yorokoglu (2009), on the welfare effect of bundling versus *à la carte* pricing of cable channels all assumed that the content quality of the channels is not affected by the price regulations. To the extent that television programming is well approximated as excludable public goods (due to its large fixed cost, but almost negligible cost of serving additional consumers), our paper cautions that forcing *à la carte* pricing *might* lead to the deterioration of the quality, or even elimination, of some channels.

The importance of use exclusion and bundling for the provision of public goods is also strikingly illustrated by the recent struggle of the newspaper industry. Traditional print newspapers are able to both exclude non-subscribers and bundle news with advertising. The recent shift to online news makes it much harder for news organizations to exclude non-subscribers from access, and it also makes it harder to bundle news with other services. To the extent that high quality journalism is non-rival, the insights in our paper would suggest that the quality of journalism is likely to decline as a result of the recent changes in the newspaper industry. Paradoxically, the removal of an exclusion technology may lead to less access to the public good, in this case, high quality journalism, for everyone.

²⁷This line of reasoning was an important part of the motivation in the decision by the Office of Fair Trading (2003) in the U.K. on alleged anti-competitive mixed bundling by the British Sky Broadcasting Limited.

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APPENDIX

PROOF OF LEMMA 4: Let j be some good for which $\theta_i^j = h$. The optimality conditions for problem (14) with respect to $\eta_i^j(\theta_i)$ are:

$$\sum_{\theta_{-i} \in \Theta_{-i}} \beta(\theta) \rho^j(\theta) h + \lambda(m) m \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta) h \quad (\text{A1})$$

$$-\lambda(m+1)(M-m) \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta) h + \gamma_i^j(\theta_i) - \phi_i^j(\theta_i) = 0;$$

$$\gamma_i^j(\theta_i) \eta_i^j(\theta_i) = 0 \text{ and } \phi_i^j(\theta_i) [1 - \eta_i^j(\theta_i)] = 0, \quad (\text{A2})$$

where $\gamma_i^j(\theta_i)$ and $\phi_i^j(\theta_i)$ are respectively the multipliers for the constraints $\eta_i^j(\theta_i) \geq 0$ and $1 - \eta_i^j(\theta_i) \geq 0$. Since by assumption $\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta_i, \theta_{-i}) > 0$ and since

$$\sum_{\theta_{-i} \in \Theta_{-i}} \beta(\theta) \rho^j(\theta) h = \beta_i(\theta_i) \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta) h, \quad (\text{A3})$$

we can simplify (A1) as:

$$\beta_i(\theta_i) h + \lambda(m) m h - \lambda(m+1)(M-m) h + \frac{\gamma_i^j(\theta_i) - \phi_i^j(\theta_i)}{\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta)} = 0. \quad (\text{A4})$$

Together with the complementary slackness conditions (A2), (A4) implies that

$$\eta_i^j(\theta_i) = \begin{cases} 1 & \text{if } \beta_i(\theta_i) + \lambda(m) m - \lambda(m+1)(M-m) > 0 \\ x \in [0, 1] & \text{if } \beta_i(\theta_i) + \lambda(m) m - \lambda(m+1)(M-m) = 0 \\ 0 & \text{if } \beta_i(\theta_i) + \lambda(m) m - \lambda(m+1)(M-m) < 0. \end{cases}$$

However, (17) implies that

$$\beta_i(\theta_i) + \lambda(m) m - \lambda(m+1)(M-m) = (1 + \Lambda) \beta_i(\theta_i) > 0.$$

Hence $\eta_i^j(\theta_i) = 1$ for all $\theta_i \in \Theta$ such that $\theta_i^j = h$. ■

PROOF OF LEMMA 5: Consider $\theta_i \in \Theta$ with $m(\theta_i) = m$ and let j be some good for which $\theta_i^j = l$. The optimality conditions for problem (14) with respect to $\eta_i^j(\theta_i)$ are:

$$\sum_{\theta_{-i} \in \Theta_{-i}} \beta(\theta) \rho^j(\theta) l + \lambda(m) m \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta) l \quad (\text{A5})$$

$$-\lambda(m+1) \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta) [(M-m-1)l + h] + \gamma_i^j(\theta_i) - \phi_i^j(\theta_i) = 0;$$

$$\gamma_i^j(\theta_i) \eta_i^j(\theta_i) = 0 \text{ and } \phi_i^j(\theta_i) (1 - \eta_i^j(\theta_i)) = 0, \quad (\text{A6})$$

where $\gamma_i^j(\theta_i)$ and $\phi_i^j(\theta_i)$ are the multipliers for the constraints $\eta_i^j(\theta_i) \geq 0$ and $1 - \eta_i^j(\theta_i) \geq 0$ respectively. Using (A3) as in the proof of Lemma 4, we may rearrange (A5) as:

$$\beta_i(\theta_i) l + \lambda(m) m l - \lambda(m+1) [(M-m-1)l + h] + \frac{\gamma_i^j(\theta_i) - \phi_i^j(\theta_i)}{\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta)} = 0. \quad (\text{A7})$$

Note that:

$$\begin{aligned}
& \beta_i(\theta_i) l + \lambda(m) ml - \lambda(m+1) [(M-m-1)l + h] \\
&= \beta_i(\theta_i) l(1+\Lambda) - \lambda(m+1)(h-l) \\
&= \beta_i(\theta_i) l(1+\Lambda) - (h-l) \frac{m!(M-(m+1))!}{M!} \Lambda \sum_{j=m+1}^M \beta_j \\
&= \frac{m!(M-(m+1))!}{M!} \left[\beta_m(M-m)l(1+\Lambda) - (h-l) \Lambda \sum_{j=m+1}^M \beta_j \right] \\
&= \frac{m!(M-(m+1))!}{M!} (1+\Lambda) G_m(\Phi)
\end{aligned} \tag{A8}$$

for $\Phi = \frac{\Lambda}{1+\Lambda}$, where the first equality uses (17), the second uses Lemma 3 and the third uses the formula (7) for β_m . Substituting (A8) into (A7) and using the complementary slackness conditions in (A6), we have

$$\eta_i^j(\theta_i) = \eta(m) = \begin{cases} 0 & \text{if } G_m(\Phi) < 0 \\ z \in [0, 1] & \text{if } G_m(\Phi) = 0 \\ 1 & \text{if } G_m(\Phi) > 0, \end{cases}$$

as asserted. \blacksquare

PROOF OF LEMMA 6: Using the notation $H^j(\theta, m)$ and $L^j(\theta, m)$ introduced in Section 3.4, we can write the optimality conditions for problem (14) with respect to $\rho^j(\theta)$ as:

$$\begin{aligned}
& \beta(\theta) \left[\sum_{i=1}^n \eta_i^j(\theta_i) \theta_i^j - cn \right] + \sum_{m=0}^M \lambda(m) m [H^j(\theta, m) \beta_{-i}(\theta_{-i}) h + L^j(\theta, m) \beta_{-i}(\theta_{-i}) \eta(m) l] \\
& - \sum_{m=0}^{M-1} \lambda(m+1) \{ H^j(\theta, m) \beta_{-i}(\theta_{-i}) (M-m) h + L^j(\theta, m) \beta_{-i}(\theta_{-i}) \eta(m) [(M-m)l + (h-l)] \} \\
& - \Lambda \beta(\theta) cn + \gamma^j(\theta) - \phi^j(\theta) = 0,
\end{aligned} \tag{A9}$$

together with the complementary slackness conditions, where $\eta(m)$ denotes the probability that an agent with m high valuation goods gets access to her low valuation goods valuations (as characterized in Lemma 5), $\gamma^j(\theta)$ and $\phi^j(\theta)$ are the multipliers associated with the boundary conditions for $\rho^j(\theta) \in [0, 1]$.²⁸ Using (A3) and the fact that

$$\sum_{i=1}^n \eta_i^j(\theta_i) \theta_i^j = \sum_{m=0}^M H^j(\theta, m) h + \sum_{m=0}^M L^j(\theta, m) \eta(m) l,$$

we can rewrite (A9), after collecting terms, as:

$$\begin{aligned}
& \sum_{m=0}^M H^j(\theta, m) \left\{ h + \frac{h}{\beta_i(\theta_i)} [\lambda(m) m - \lambda(m+1) (M-m)] \right\} \\
& \sum_{m=0}^M L^j(\theta, m) \left\{ \eta(m) l + \frac{\eta(m)}{\beta_i(\theta_i)} \{ \lambda(m) ml - \lambda(m+1) [(M-m)l + (h-l)] \} \right\} - (1+\Lambda) cn + \frac{\gamma^j(\theta) - \phi^j(\theta)}{\beta(\theta)} = 0,
\end{aligned}$$

which we can further simplify this further, after using (17), to:

$$\sum_{m=0}^M H^j(\theta, m) (1+\Lambda) h + \sum_{m=0}^M L^j(\theta, m) \left\{ \eta(m) l + \eta(m) \left[\Lambda - \frac{\lambda(m+1)}{\beta_i(\theta_i)} (h-l) \right] \right\} - (1+\Lambda) cn + \frac{\gamma^j(\theta) - \phi^j(\theta)}{\beta(\theta)} = 0. \tag{A10}$$

²⁸The notation is somewhat unsatisfactory in that $\beta_{-i}(\theta_{-i})$ would be more appropriately denoted by $\beta_{-i}(\theta|\theta_i)$ where θ_i would describe a type that would enter in the particular term (and therefore change with each term). Concretely, $H^j(\theta, m) \beta_{-i}(\theta_{-i})$ would correspond to $\beta_{-i}(\theta|\theta_i)$ for θ_i with $\theta_i^j = h$ and a total of m high valuations.

From (16) and (7), we have:

$$\frac{\lambda(m+1)}{\beta_i(\theta_i)} = \frac{1}{\beta_m(M-m)} \Lambda \sum_{j=m+1}^M \beta_j. \quad (\text{A11})$$

Hence,

$$\begin{aligned} & \eta(m)l + \eta(m) \left[\Lambda l - \frac{\lambda(m+1)}{\beta_i(\theta_i)} (h-l) \right] = \eta(m)l + \eta(m) \left[\Lambda l - \frac{1}{\beta_m(M-m)} \Lambda \sum_{j=m+1}^M \beta_j (h-l) \right] \\ &= \frac{(1+\Lambda)\eta(m)}{\beta_m(M-m)} \left\{ (1-\Phi)\beta_m(M-m)l + \Phi \left[\beta_m(M-m)l - \sum_{j=m+1}^M \beta_j (h-l) \right] \right\} \\ &= \frac{(1+\Lambda)}{\beta_m(M-m)} \max\{0, G_m(\Phi)\}. \end{aligned}$$

Substituting this into (A10) gives us

$$(1+\Lambda) \left[\sum_{m=0}^M H^j(\theta, m) h + \sum_{m=0}^M L^j(\theta, m) \left\{ \frac{1}{\beta_m(M-m)} \max\{0, G_m(\Phi)\} \right\} - cn \right] + \frac{\gamma^j(\theta) - \phi^j(\theta)}{\beta(\theta)} = 0,$$

which, together with the complementary slackness conditions, gives the desired result. \blacksquare

PROOF OF LEMMA 8: Since by assumption (ρ, η, t) is not *ex post* efficient, it must be the case that $(n-1)h > nc$. Otherwise, the *ex post* optimal mechanism is either to provide the public goods if and only if $\theta_i^j = h$ for all i when $(n-1)h < nc < nh$, or to never provide any of the goods when $nh \leq nc$; in either case, the *ex post* optimal mechanism is trivially implementable under the constraints in (14). Thus we assume that $(n-1)h > nc$ in the remainder of the proof.

Suppose that a downward adjacent incentive constraint does not bind, then there exists some θ_i such that

$$\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta) \eta_i^j(\theta_i) \theta_i^j - t_i(\theta_i) > \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta_{-i}, \theta_i | l_k) \eta_i^j(\theta_i | l_k) \theta_i^j - t_i(\theta_i | l_k).$$

If $\eta_i^j(\theta_i | l_k) < 1$, we can increase $\eta_i^j(\theta_i | l_k)$ slightly without upsetting the constraint. Since $(n-1)h > nc$, it follows that $\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^j(\theta_{-i}, \theta_i | l_k) > 0$ for every j . Hence, an increase in $\eta_i^j(\theta_i | l_k)$ increases the value of the objective function, contradicting the postulated optimality of (ρ, η, t) . Suppose instead that $\eta_i^j(\theta_i | l_k) = 1$. Then, we can find some $\varepsilon > 0$ and raise the tax to $\tilde{t}_i(\theta_i) = t_i(\theta_i) + \varepsilon$ for θ_i and every θ_i^j with $m(\theta_i^j) = m(\theta_i)$. This raises some extra revenue. We can then proceed inductively as follows. First consider types θ_i^A with $m(\theta_i^A) = m(\theta_i) - 2$. If $\eta_i^j(\theta_i^A) < 1$ we can increase the surplus by increasing $\eta_i^j(\theta_i^A)$ for θ_i^A with $m(\theta_i^A) = m(\theta_i) - 2$ and simultaneously decrease $t_i(\theta_i^B)$ for types with $m(\theta_i^B) = m(\theta_i) - 1$ so as to keep the downward incentive constraint for types with θ_i^B . If $\eta_i^j(\theta_i^A) = 1$ and $m(\theta_i) - 2 \geq 1$ repeat the argument for types θ_i^C with $m(\theta_i^C) = m(\theta_i) - 3$. The resulting increase in access probabilities is always desirable from the social welfare viewpoint. Thus by induction, we conclude that either there exists a feasible allocation with a higher surplus or that $\eta_i^j(\theta_i) = 1$ for all θ_i and j .

However, if $\eta_i^j(\theta_i) = 1$ for all θ_i and j , i.e., if there are no active use of exclusions in the solution, then the extra revenue from $\tilde{t}_i(\theta_i)$ could be used to improve the surplus by increasing $\rho^j(\theta)$ for some θ , since the postulated mechanism (ρ, η, t) is not *ex post* efficient.

The result that the participation constraint for type-I must bind can be proved analogously. \blacksquare

PROOF OF PROPOSITION 4: First note that (21) is irrelevant when $m = M$. Moreover, it is immediate from (24) that the probability of provision given $(\mathbf{h}, \theta_{-i})$ is always weakly higher than under (θ_i, θ_{-i}) for any $\theta_{-i} \in \Theta_{-i}$. Finally, from (21) and (24), we know that a sufficient condition for monotonicity is that $\frac{1}{\beta_m(M-m)} \max\{0, G_m(\Phi)\}$ is increasing in m on $\{0, \dots, M-1\}$. But,

$$\frac{G_m(\Phi)}{\beta_m(M-m)} = l - \Phi \left[\frac{(h-l)}{\beta_m(M-m)} \sum_{j=m+1}^M \beta_j \right]. \quad \blacksquare \quad (\text{A12})$$

PROOF OF OBSERVATION 1: If valuations are independent across goods, say the probability that an agent has a high valuation for any good j is α , then the probability that an agent has k high valuations is given by $\beta_k = \frac{M!}{k!(M-k)!} \alpha^k (1-\alpha)^{M-k}$. Thus,

$$\beta_{k+1} = \frac{M!}{(k+1)!(M-k-1)!} \alpha^{k+1} (1-\alpha)^{M-k-1} = \frac{(M-k)\alpha}{(k+1)(1-\alpha)} \beta_k. \quad (\text{A13})$$

Using (A13), we have that

$$\begin{aligned} & \frac{1}{\beta_{m+1}(M-m-1)} \sum_{j=m+2}^M \beta_j = \frac{1}{\beta_{m+1}(M-m-1)} \sum_{j=m+1}^{M-1} \beta_{j+1} \\ &= \frac{(m+1)(1-\alpha)}{(M-m)\alpha(M-m-1)\beta_m} \sum_{j=m+1}^{M-1} \frac{(M-j)\alpha}{(j+1)(1-\alpha)} \beta_j \\ &= \frac{1}{(M-m)\beta_m} \sum_{j=m+1}^{M-1} \frac{(m+1)(M-j)}{(j+1)(M-m-1)} \beta_j < \frac{1}{(M-m)\beta_m} \sum_{j=m+1}^M \beta_j. \end{aligned}$$

where the inequality follows from the fact that $j \geq m+1$ and $\beta_M > 0$. ■

PROOF of COROLLARY 1: We need to verify that $G_{m+1}(\Phi) > 0$ is implied by $G_m(\Phi) \geq 0$, and that $G_{m-1}(\Phi) < 0$ is implied by $G_m(\Phi) \leq 0$. Both are immediate from (A12). ■

PROOF OF PROPOSITION 5: [Part 1(a)] Suppose, for a contradiction, that there is a subsequence of *feasible* mechanisms with $\lim_{n \rightarrow \infty} \mathbb{E}[\rho_n^j(\theta)] = \rho^*$ and $\lim_{n \rightarrow \infty} \eta_n(\tilde{m}) = \eta^*$, where \tilde{m} is the threshold number of high valuation goods in the sense of Lemma 1, which without loss of generality is taken to be constant along the sequence by the choice of a converging subsequence. We introduce the following notations:

- $\mathbb{E}[\rho_n^j(\theta) | m, \theta_i^j = h]$ denotes the conditional provision probability for good j perceived by agent i given that $\theta_i^j = h$ and $\theta_i^k = h$ with $k \in \mathcal{J}$ for exactly m goods (including j).
- $\mathbb{E}[\rho_n^k(\theta) | m, \theta_i^k = l]$ denotes the conditional provision probability for good j perceived by agent i given that $\theta_i^k = l$ and $\theta_i^j = h$ with $j \in \mathcal{J}$ for exactly m goods.

Notice that neither of the conditional probabilities above depends on j by Proposition 2. Propositions 1 and 2 jointly imply that for every $m \in \{0, \dots, M\}$, there exists some $t_n(m)$ such that the transfer $t_n(m)$ is paid by every agent i with m high valuations. The utility from truth-telling for an agent with m high valuation goods is,

$$\begin{aligned} & mh\mathbb{E}[\rho_n^j(\theta) | m, \theta_i^j = h] - t_n(m) && \text{if } m < \tilde{m} \\ & \tilde{m}h\mathbb{E}[\rho_n^j(\theta) | \tilde{m}, \theta_i^j = h] + (M - \tilde{m})\eta_n(\tilde{m})\mathbb{E}[\rho_n^k(\theta) | \tilde{m}, \theta_i^k = l] l - t_n(\tilde{m}) && \text{if } m = \tilde{m} \\ & mh\mathbb{E}[\rho_n^j(\theta) | m, \theta_i^j = h] + (M - m)\mathbb{E}[\rho_n^k(\theta) | m, \theta_i^k = l] l - t_n(m) && \text{if } m > \tilde{m}. \end{aligned}$$

Hence, the participation constraints for type- θ_i agents with $m(\theta_i) = m < \tilde{m}$ high valuations imply that:²⁹

$$t_n(m) \leq mh\mathbb{E}[\rho_n^j(\theta) | m, \theta_i^j = h] \text{ if } m(\theta_i) = m < \tilde{m}. \quad (\text{A14})$$

Similarly, the participation constraint for agents with \tilde{m} high valuations implies that:

$$t_n(\tilde{m}) \leq \tilde{m}h\mathbb{E}[\rho_n^j(\theta) | \tilde{m}, \theta_i^j = h] + (M - \tilde{m})\eta_n(\tilde{m})\mathbb{E}[\rho_n^k(\theta) | \tilde{m}, \theta_i^k = l] l \text{ if } m(\theta_i) = \tilde{m}. \quad (\text{A15})$$

²⁹Strictly speaking, we only impose the participation constraint on type-1. However, the downward adjacent incentive constraints together with the participation constraint for type-1 imply that all participation constraints hold.

The downward adjacent incentive constraint for an agent with type θ_i with $m(\theta_i) = \tilde{m} + 1$ against mis-reporting as type $\hat{\theta}_i$ (which differs from θ_i only in that one of the high valuations in θ_i is reported as low valuation) requires that

$$\begin{aligned} & (\tilde{m} + 1)h\mathbb{E} \left[\rho_n^j(\theta) \mid \tilde{m} + 1, \theta_i^j = h \right] + [M - (\tilde{m} + 1)]\mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m} + 1, \theta_i^k = l \right] l - t_n(\tilde{m} + 1) \\ & \geq \tilde{m}h\mathbb{E} \left[\rho_n^j(\theta) \mid \tilde{m}, \hat{\theta}_i^j = h \right] + \eta_n(\tilde{m})\mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m}, \hat{\theta}_i^k = l \right] \{[M - (\tilde{m} + 1)]l + h\} - t_n(\tilde{m}) \\ & \geq \eta_n(\tilde{m})\mathbb{E} \left[\rho_n^j(\theta) \mid \tilde{m}, \theta_i^k = l \right] (h - l), \end{aligned}$$

where the first inequality is required by the downward adjacent incentive constraint, and the second inequality follows from (A15). Thus,

$$\begin{aligned} t_n(\tilde{m} + 1) & \leq (\tilde{m} + 1)h\mathbb{E} \left[\rho_n^j(\theta) \mid \tilde{m} + 1, \theta_i^j = h \right] + (M - \tilde{m} - 1)\mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m} + 1, \theta_i^k = l \right] l \\ & \quad - \eta_n(\tilde{m})\mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m}, \theta_i^k = l \right] (h - l). \end{aligned} \quad (\text{A16})$$

Finally, for every type θ_i with $m(\theta_i) = m > \tilde{m} + 1$, the incentive constraint that type θ_i will not find it profitable to mis-report as type $\hat{\theta}_i$ with $m(\hat{\theta}_i) = \tilde{m} + 1$ implies that

$$\begin{aligned} & mh\mathbb{E} \left[\rho_n^j(\theta) \mid m, \theta_i^j = h \right] + (M - m)\mathbb{E} \left[\rho_n^k(\theta) \mid m, \theta_i^k = l \right] l - t_n(m) \\ & \geq (\tilde{m} + 1)h\mathbb{E} \left[\rho_n^j(\theta) \mid \tilde{m} + 1, \hat{\theta}_i^j = h \right] + \mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m} + 1, \hat{\theta}_i^k = l \right] [(M - m)l + (m - \tilde{m} - 1)h] - t_n(\tilde{m} + 1) \\ & \geq \mathbb{E} \left[\rho_n^j(\theta) \mid \tilde{m} + 1, \theta_i^j = l \right] [(m - \tilde{m} - 1)(h - l)] + \eta_n(\tilde{m})\mathbb{E} \left[\rho_n^j(\theta) \mid \tilde{m}, \theta_i^k = l \right] (h - l), \end{aligned}$$

where the last inequality follows from (A16). Hence, for all $m > \tilde{m} + 1$,

$$\begin{aligned} t_n(m) & \leq mh\mathbb{E} \left[\rho_n^j(\theta) \mid m, \theta_i^j = h \right] + (M - m)\mathbb{E} \left[\rho_n^j(\theta) \mid m, \theta_i^k = l \right] l \\ & \quad - \mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m} + 1, \theta_i^k = l \right] (m - \tilde{m} - 1)(h - l) - \eta_n(\tilde{m})\mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m}, \theta_i^k = l \right] (h - l). \end{aligned} \quad (\text{A17})$$

From the ‘‘Paradox of Voting’’-like Lemma 9, we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\rho_n^j(\theta) \mid m, \theta_i^j = h \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\rho_n^k(\theta) \mid m, \theta_i^k = l \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\rho_n^k(\theta) \mid \tilde{m}, \theta_i^k = l \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\rho_n^j(\theta) \right] = \rho^*. \quad (\text{A18})$$

Combining (A18) with (A14)-(A17), it follows that, for every $\varepsilon > 0$ there is some N such that when $n \geq N$,

$$t_n(m) \leq \begin{cases} \rho^*mh + \varepsilon & \text{if } m < \tilde{m} \\ \rho^*[\tilde{m}h + (M - \tilde{m})\eta^*l] + \varepsilon & \text{if } m = \tilde{m} \\ \rho^*[(\tilde{m} + 1)h + (M - \tilde{m} - 1)l - \eta^*(h - l)] + \varepsilon & \text{if } m \geq \tilde{m} + 1. \end{cases}$$

Now summing over m , we find that the expected per capita transfer revenue must satisfy:

$$\begin{aligned} \sum_{m=0}^M \beta_m t_n(m) & \leq \rho^* \left[h \sum_{m=0}^{\tilde{m}} \beta_m m + \left(\sum_{m=\tilde{m}+1}^M \beta_m \right) \{(\tilde{m} + 1)h + (M - \tilde{m} - 1)l\} \right] \\ & \quad + \rho^*\eta^* \left[\beta_{\tilde{m}}(M - \tilde{m})l - \left(\sum_{m=\tilde{m}+1}^M \beta_m \right) (h - l) \right] + \varepsilon \\ & = \rho^*(1 - \eta^*) \left\{ \left(\sum_{m=\tilde{m}+1}^M \beta_m \right) [(\tilde{m} + 1)h + (M - \tilde{m} - 1)l] + h \sum_{m=0}^{\tilde{m}} \beta_m m \right\} \\ & \quad + \rho^*\eta^* \left\{ \left(\sum_{m=\tilde{m}}^M \beta_m \right) [\tilde{m}h + (M - \tilde{m})l] + h \sum_{m=0}^{\tilde{m}-1} \beta_m m \right\} + \varepsilon \\ & = \rho^* [(1 - \eta^*)R(\tilde{m} + 1) + \eta^*R(\tilde{m})] + \varepsilon. \end{aligned} \quad (\text{A19})$$

Moreover, for any $\varepsilon > 0$, we can find N such that when $n > N$, $M \sum_{\theta \in \Theta^n} \beta(\theta) \rho_n^j(\theta) c \geq \rho^* M c - \varepsilon$. Together with (A19), this implies that

$$\begin{aligned} \sum_{i=1}^n \sum_{\theta_i \in \Theta} \beta_i(\theta_i) t_{in}(\theta_i) - \sum_{\theta \in \Theta^n} \beta(\theta) \sum_{j=1}^M \rho_n^j(\theta) c n &= n \sum_{m=0}^M \beta_m t_n(m) - M \sum_{\theta \in \Theta^n} \beta(\theta) \rho_n^j(\theta) c \\ &\leq \rho^* [(1 - \eta^*) R(\tilde{m} + 1) + \eta^* R(\tilde{m}) - cM] + 2\varepsilon. \end{aligned} \quad (\text{A20})$$

Under the hypothesis that $\max_m R(m) < cM$, we know that $(1 - \eta^*) R(\tilde{m} + 1) + \eta^* R(\tilde{m}) - cM < 0$. Thus, for any $\rho^* > 0$, there exists $\varepsilon > 0$ such that the right hand side of (A20) is negative. Hence, the budget constraint (11) is violated for large n along any sequence with positive provision probability. Thus $\lim_{n \rightarrow \infty} \mathbb{E}[\rho_n^j(\theta)] = 0$ for any convergent subsequence.

[Part 1(b) and Part 2] Let m^* be the smallest m such that $R(m) - cM > 0$. Consider the sequence of mechanisms $(\hat{\rho}_n, \hat{\eta}_n, \hat{t}_n)$ where $\hat{\rho}_n^j(\theta) = 1$ for every n and θ , where all agents are given access to their high valuation goods and the inclusion rule for low valuation goods is

$$\hat{\eta}_n(m) = \begin{cases} 0 & \text{if } m \leq m^* - 2 \\ \eta^* = \frac{R(m^*) - cM}{R(m^*) - R(m^* - 1)} & \text{if } m = m^* - 1 \\ 1 & \text{if } m \geq m^*, \end{cases}$$

and where the transfer rule is

$$\hat{t}_n(m) = \begin{cases} mh & \text{if } m \leq m^* - 2 \\ (m^* - 1)h + (M - m^* + 1)\eta^*l & \text{if } m = m^* - 1 \\ m^*h + (M - m^*)l - \eta^*(h - l) & \text{if } m \geq m^*. \end{cases}$$

It is easy to check that under the above mechanism truth-telling is incentive compatible and individually rational. It can also be verified that the per capita expected transfer under the above mechanism is

$$\begin{aligned} \sum_{m=0}^M \beta_m \hat{t}_n(m) &= h \sum_{m=0}^{m^*-2} \beta_m m + \beta_{m^*-1} [(m^* - 1)h + (M - m^* + 1)\eta^*l] + \left(\sum_{m=m^*}^M \beta_m \right) [m^*h + (M - m^*)l - \eta^*(h - l)] \\ &= (1 - \eta^*) R(m^* + 1) + \eta^* R(m^*) = cM, \end{aligned} \quad (\text{A21})$$

thus the balanced budget constraint also exactly holds. Hence $(\hat{\rho}_n, \hat{\eta}_n, \hat{t}_n)$ is incentive feasible for every n . The associated *per capita* social surplus with the above mechanism is

$$\frac{S_n^*(\theta)}{n} = \sum_{j=1}^M \left\{ h \sum_{m=1}^M \frac{H^j(\theta, m)}{n} + l \left[\eta^* \frac{L^j(\theta, m^*)}{n} + \sum_{m=m^*}^M \frac{L^j(\theta, m)}{n} \right] \right\} - cM, \quad (\text{A22})$$

where the notations $H^j(\theta, m)$ and $L^j(\theta, m)$ were explained in Section 3.4. Noting that $\lim_{n \rightarrow \infty} \frac{H^j(\theta, m)}{n} = \frac{m}{M} \beta_m$ and $\lim_{n \rightarrow \infty} \frac{L^j(\theta, m)}{n} = \frac{M-m}{M} \beta_m$, we have

$$\lim_{n \rightarrow \infty} \frac{S_n^*(\theta)}{n} = h \sum_{m=1}^M m \beta_m + l \left[\eta^* (M - m^*) \beta_{m^*} + \sum_{m=m^*+1}^M (M - m) \beta_m \right] - cM > \varepsilon = 0 \quad (\text{A23})$$

Now we show that, as $n \rightarrow \infty$, there is no other incentive feasible mechanism that yields higher per capita social surplus than the above mechanism. Invoking Corollary 1, suppose that the surplus maximizing mechanism is characterized by an inclusion rule with threshold \tilde{m}^* and inclusion probability $\tilde{\eta}^*$ given \tilde{m}^* , and a provision rule $\tilde{\rho}_n^j$ that must be characterized by (24).

Since by definition m^* is the smallest m such that $R(m) - cM > 0$, we know that there are three possibilities: (1). $R(\tilde{m}^*) - cM < 0$; (2). $R(\tilde{m}^*) - cM > 0$ but $\tilde{m}^* > m^*$; or (3). $\tilde{m}^* = m^*$. If $R(\tilde{m}^*) - cM < 0$, then an argument identical with that in the proof of Part 1(a) establishes that $\lim_{n \rightarrow \infty} \mathbb{E}[\tilde{\rho}_n^j(\theta)] \rightarrow 0$; thus the social welfare is lower than that obtained from $(\hat{\rho}_n, \hat{\eta}_n, \hat{t}_n)$. If $\tilde{m}^* > m^*$ but $R(\tilde{m}^*) - cM > 0$, then $\lim_{n \rightarrow \infty} \mathbb{E}[\rho_n^j(\theta)] \rightarrow$

1. The social surplus must therefore converge towards the same expression as in (A23), but with m^* replaced by \tilde{m}^* . But, if $\tilde{m}^* > m^*$, the social surplus is smaller because there is more exclusion. Finally, suppose that $\tilde{m}^* = m^*$. If $\hat{\eta}_n(m^*) \rightarrow \tilde{\eta}^* > \eta^*$ an argument identical with Part 1(a) establishes that $\lim_{n \rightarrow \infty} \mathbb{E}[\tilde{\rho}_n^j(\theta)] \rightarrow 0$; if $\hat{\eta}_n(m^*) \rightarrow \tilde{\eta}^* < \eta^*$ we have that $\lim_{n \rightarrow \infty} \mathbb{E}[\tilde{\rho}_n^j(\theta)] \rightarrow 1$ and again social surplus converges towards the same expression as in (A23), but with η^* replaced by $\tilde{\eta}^*$. Again, the social surplus is smaller if $\tilde{\eta}^* < \eta^*$ since there is more exclusion. ■

PROOF OF PROPOSITION 6: To prove Proposition 6, we add the (non-adjacent) constraint that the highest type should not have an incentive to pretend to be the lowest type:

$$0 \leq \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta_{-i}, \mathbf{h}) h - t_i(\mathbf{h}) - \left[\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^M \rho^j(\theta_{-i}, \mathbf{1}) \eta_i^j(\mathbf{1}) h - t_i(\mathbf{1}) \right] \quad (\text{A24})$$

to program (14). Let ψ denote the multiplier on constraint (A24). Similar to (17), with the additional constraint (A24) the optimality conditions with respect to $t(\theta_i)$ for types with 2, 1 and 0 high valuation goods are, respectively,

$$\begin{aligned} -2\lambda(2) - \psi + \beta_2\Lambda &= 0 \\ \lambda(2) - \lambda(1) + \frac{1}{2}\beta_1\Lambda &= 0 \\ 2\lambda(1) + \psi - \lambda(0) + \beta_0\Lambda &= 0. \end{aligned}$$

We now proceed with a sequence of intermediate results.

Claim A1 *Suppose that $\frac{\beta_1}{2} < \frac{\beta_0\beta_2}{1-\beta_0}$. Then (A24) binds.*

To see this, suppose that the additional constraint (A24) does not bind. Then we know the solution with constraint (A24) is unchanged from (14). The inclusion rule (21) for the special case with $M = 2$ is simplified to:

$$\eta(1) \equiv \begin{cases} 0 & \text{if } G_1(\Phi) < 0 \\ z \in [0, 1] & \text{if } G_1(\Phi) = 0 \\ 1 & \text{if } G_1(\Phi) > 0, \end{cases} \quad \eta(0) \equiv \begin{cases} 0 & \text{if } G_0(\Phi) < 0 \\ z \in [0, 1] & \text{if } G_0(\Phi) = 0 \\ 1 & \text{if } G_0(\Phi) > 0, \end{cases}$$

and the provision rule (24) is simplified to:

$$\rho^j(\theta) = \begin{cases} 0 & \text{if } [H^j(\theta, 1) + H^j(\theta, 2)]h + \frac{L^j(\theta, 0)}{2\beta_0} \max\{0, G_0(\Phi)\} + \frac{L^j(\theta, 1)}{\beta_1} \max\{0, G_1(\Phi)\} - cn < 0 \\ z \in [0, 1] & \text{if } [H^j(\theta, 1) + H^j(\theta, 2)]h + \frac{L^j(\theta, 0)}{2\beta_0} \max\{0, G_0(\Phi)\} + \frac{L^j(\theta, 1)}{\beta_1} \max\{0, G_1(\Phi)\} - cn = 0 \\ 1 & \text{if } [H^j(\theta, 1) + H^j(\theta, 2)]h + \frac{L^j(\theta, 0)}{2\beta_0} \max\{0, G_0(\Phi)\} + \frac{L^j(\theta, 1)}{\beta_1} \max\{0, G_1(\Phi)\} - cn > 0, \end{cases}$$

where

$$G_1(\Phi) = \beta_1 l - \Phi(h-l)\beta_2; \quad G_0(\Phi) = \beta_0 2l - \Phi(h-l)(\beta_1 + \beta_2).$$

Notice that

$$\begin{aligned} \frac{G_1(\Phi)}{\beta_1} &= l - \frac{\Phi(h-l)\beta_2}{\beta_1} \\ &< l - \frac{\Phi(h-l)\beta_2}{2\frac{\beta_0\beta_2}{1-\beta_0}} = l - \frac{\Phi(h-l)(\beta_1 + \beta_2)}{2\beta_0} = \frac{G_0(\Phi)}{2\beta_0}. \end{aligned}$$

Thus, we must have $\eta(1) \leq \eta(0)$ and $\rho^j(\theta_{-i}, \theta_i) \leq \rho^j(\theta_{-i}, ll)$ if $\theta_i^j = l$ and $\theta_i^k = h$ for $k \neq j$.

Let $t(0)$, $t(1)$ and $t(2)$ denote transfers for types with 0, 1 and 2 high valuation goods respectively. Now we show that type- (h, h) will have strictly incentive to mis-report as type- (l, l) , a contradiction. To see this, recall from Lemma 8 that in the solution to the original problem (14), the downward adjacent incentive constraints for type- (h, h) and types (h, l) or (l, h) must bind if the postulated mechanism is not ex post efficient (which is clearly

the case). Let $U(\tilde{\theta}_i|\theta_i)$ denote the expected utility for type- θ_i agent from truth-telling and from mis-reporting as type- $\tilde{\theta}_i$. We have:

$$\begin{aligned}
U(hh|hh) &= \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, hh) + \rho^2(\theta_{-i}, hh)] h - t(2) \\
&= \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, hl) h + \rho^2(\theta_{-i}, hl) \eta(1) h] - t(1) \\
&= \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, ll) \eta(0) h + \rho^2(\theta_{-i}, ll) \eta(0) l] - t(0) + \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^2(\theta_{-i}, hl) \eta(1) (h - l) \\
&= \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, ll) \eta(0) h + \rho^2(\theta_{-i}, ll) \eta(0) h] - t(0) \\
&\quad + \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^2(\theta_{-i}, hl) \eta(1) - \rho^2(\theta_{-i}, ll) \eta(0)] (h - l) \\
&\leq U(ll|hh)
\end{aligned}$$

where the first inequality follows from binding downward adjacent incentive constraint for type- (h, h) , the second equality follows from binding downward adjacent incentive constraint for type- (h, l) , the third equality follows from re-arranging and the last inequality follows from the result above that $\eta(1) \leq \eta(0)$ and $\rho^j(\theta_{-i}, \theta_i) \leq \rho^j(\theta_{-i}, ll)$. Thus, constraint (A24) binds or is violated, a contradiction.

Claim A2 Suppose that $\frac{\beta_1}{2} < \frac{\beta_0 \beta_2}{1 - \beta_0}$. Then the downward adjacent constraints for type- (h, h) and type- (h, l) or type- (l, h) must both bind in the optimal solution.

To see this, first note that it could not be the case that both the downward adjacent constraints for type- (h, h) and type- (h, l) are slack in the optimal solution. If so, then it will immediately follow that type- (h, h) will have strict incentive to mis-report as type- (l, l) , contradicting Claim A1. So suppose that the downward adjacent incentive constraint for type- (h, l) [or type- (l, h)] binds, but the downward adjacent incentive constraint for type- (h, h) is slack, i.e.,

$$\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, hh) + \rho^2(\theta_{-i}, hh)] h - t(2) > \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, hl) h + \rho^2(\theta_{-i}, hl) \eta(1) h] - t(1). \tag{A25}$$

Given (A25) and a binding (A24), the optimality conditions with respect to $\eta_i^2(hl)$ are the complementary slackness conditions together with the first order condition:

$$\sum_{\theta_{-i} \in \Theta_{-i}} \beta(\theta_{-i}, hl) \rho^2(\theta_{-i}, hl) l + \lambda(1) \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^2(\theta_{-i}, hl) l + \gamma_i^2(hl) - \phi_i^2(hl) = 0,$$

where $\gamma_i^2(hl)$ and $\phi_i^2(hl)$ are the multipliers associated with $\eta_i^2(hl) \geq 0$ and $\eta_i^2(hl) \leq 1$ respectively. This immediately implies that $\eta_i^2(hl) = \eta_i^1(lh) = \eta(1) = 1$ in optimum.

Similarly, given (A25) and a binding (A24) the optimality conditions with respect to $\rho^1(\theta)$ are the complementary slackness conditions together with the first order condition

$$\begin{aligned}
&\{h [H^1(\theta, 1) + H^1(\theta, 2)] + l [L^1(\theta, 1) + \eta(0) L^1(\theta, 0)] - cn\} + \lambda(1) \frac{2}{\beta_1} [H^1(\theta, 1) h + L^1(\theta, 1) \eta(0) l] \\
&+ \psi \frac{1}{\beta_2} H^1(\theta, 2) h - \psi \frac{1}{\beta_0} \beta_{-i}(\theta_{-i}) \eta(0) H^1(\theta, 2) - \lambda(1) \frac{\eta(0)}{\beta_0} [H^1(\theta, 1) h + L^1(\theta, 1) l] + \lambda(0) \frac{1}{\beta_0} L^1(\theta, 0) l - \Lambda cn \\
&+ \frac{\gamma^1(\theta) - \phi^1(\theta)}{\beta(\theta)} = 0, \tag{A26}
\end{aligned}$$

where $\gamma^1(\theta)$ and $\phi^1(\theta)$ are respectively the multipliers associated with $\rho^1(\theta) \geq 0$ and $1 - \rho^1(\theta) \geq 0$. Now consider the first order condition (A26) for profile $\theta' = (\theta_{-i}, lh)$ and $\theta'' = (\theta_{-i}, ll)$ for any $\theta_{-i} \in \Theta^{n-1}$, we have $L^1(\theta', 0) =$

$L^1(\theta'', 0) - 1$, $H^1(\theta', 1) = H^1(\theta'', 1)$, $L^1(\theta', 1) = L^1(\theta'', 1) + 1$, $H^1(\theta', 2) = H^1(\theta'', 2)$. Thus the terms in the first two lines in (A26) is higher under θ' . Thus it must be the case that $\rho^1(\theta') \geq \rho^1(\theta'')$, i.e., $\rho^1(\theta_{-i}, hl) \geq \rho^1(\theta_{-i}, ll)$ for every θ_{-i} . Together with the fact that $\eta(1) = 1 \geq \eta(0)$ we established above, we thus have:

$$\sum_{\theta_{-i} \in \Theta_{-i}} \beta(\theta_{-i}) \rho^1(\theta_{-i}, lh) \eta(1) \geq \sum_{\theta_{-i} \in \Theta_{-i}} \beta(\theta_{-i}) \rho^1(\theta_{-i}, ll) \eta(0), \quad (\text{A27})$$

i.e., the perceived probability to consume a low valuation good is weakly higher for a consumer with a high valuation for the other good. Hence,

$$\begin{aligned} U(hh|hh) &= \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, hh) + \rho^2(\theta_{-i}, hh)] h - t(2) \\ &= \overbrace{\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, ll) \eta(0) h + \rho^2(\theta_{-i}, ll) \eta(0) h]}^{U(ll|hh)} - t(0) \\ &= \overbrace{\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, ll) \eta(0) l + \rho^2(\theta_{-i}, ll) \eta(0) h]}^{U(ll|lh)} - t(0) + (h-l) \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^1(\theta_{-i}, ll) \eta(0) \\ &= \overbrace{\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, lh) \eta(1) l + \rho^2(\theta_{-i}, lh) h]}^{U(lh|lh)} - t(1) + (h-l) \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^1(\theta_{-i}, ll) \eta(0) \\ &= \overbrace{\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) [\rho^1(\theta_{-i}, lh) \eta(1) h + \rho^2(\theta_{-i}, lh) h]}^{U(lh|hh)} - t(1) \\ &\quad + (h-l) \left[\sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^1(\theta_{-i}, ll) \eta(0) - \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i}) \rho^1(\theta_{-i}, lh) \eta(1) \right] \\ &\leq U(lh|hh), \end{aligned} \quad (\text{A28})$$

where the second equality follows from the binding (A24), third and fifth equality are simply re-arrangement of terms, and the fourth equality follows from the postulated downward adjacent incentive constraint for type- (l, h) and the last inequality follows from (A27). Inequality (A28) contradicts the postulated (A25).

The statement that the downward adjacent incentive constraint for type- (h, l) , or type- (l, h) must bind can be proved analogously.

We thus conclude from Claims A1 and A2 that all the incentive constraints bind for this case. Examining these incentive constraints and using the key ‘‘Paradox of Voting’’-like Lemma 9, we have that in the limit as $n \rightarrow \infty$, these constraints imply:

$$\begin{aligned} 2h - t(2) &= h + \eta(1)h - t(1) \\ 2h - t(2) &= 2h\eta(0) - t(0) \\ h + \eta(1)l - t(1) &= (h+l)\eta(0) - t(0) \end{aligned}$$

where the first equality is the limit of the binding constraint (A24) and the second is the limit of the binding downward adjacent incentive constraints (15) for types (h, h) and (h, l) respectively. These three equalities immediately imply that in the limit as $n \rightarrow \infty$, $\eta(1) = \eta(0)$. This inclusion rule immediately implies that bundling is not used in the optimal mechanism in the limit (as well as when n is sufficiently large). We thus conclude that the limit provision and inclusion rules for this case is identical to the single good case analyzed in Section 4.1. The desired results follow. \blacksquare