ONLINE APPENDIX: OMITTED PROOFS FOR RESULTS IN
OPTIMAL PROVISION OF MULTIPLE EXCLUDABLE PUBLIC GOODS
FANG AND NORMAN (2009)

PROOF OF PROPOSITION 1: The proposition follows from the following Claims B1 and B2 below.

Claim B1 *For any incentive feasible mechanism $\mathcal{G}$ of the form (3), there exist an incentive feasible mechanism*

$$G=\left(\left(\rho^{j},\eta_{1}^{j},...,\eta_{n}^{j}\right)_{j\in\mathcal{J}},\left(t_{i}\right)_{i\in\mathcal{I}}\right), \tag{B1}$$

that generates the same social surplus, where $\rho^{j}:\Theta^{n}\rightarrow[0,1]$ is the provision rule for good j, $\eta_{i}^{j}:\Theta\rightarrow[0,1]$ is the inclusion rule for agent i and good j,and $t_{i}:\Theta\rightarrow R$ is the transfer rule for agent i.

To see this, consider an incentive feasible mechanism $\mathcal{G}$. Pick any $k\in[0,1]$ and define, for each $\theta\in\Theta^{n}, j\in\mathcal{J}$ and $i\in\mathcal{I}$,

$$\begin{aligned}
\rho^{j}\left(\theta\right) &= \mathrm{E}_{\Xi}\zeta^{j}\left(\theta,\vartheta\right)=\int_{0}^{1}\zeta^{j}\left(\theta,\vartheta\right)d\vartheta && \text{(B2)}\\
\eta_{i}^{j}\left(\theta_{i}\right) &= \begin{cases}\frac{\mathrm{E}_{-i}\zeta^{j}(\theta,\vartheta)\omega_{i}^{j}(\theta,\vartheta)}{\mathrm{E}_{-i}\zeta^{j}(\theta,\vartheta)}=\frac{\int_{\Theta_{-i}^{n}}\int_{0}^{1}\zeta^{j}(\theta,\vartheta)\omega_{i}^{j}(\theta,\vartheta)d\vartheta d\mathbf{F}(\theta_{-i})}{\int_{\Theta_{-i}^{n}}\int_{0}^{1}\zeta^{j}(\theta,\vartheta)d\vartheta d\mathbf{F}(\theta_{-i})} & \text{if } \int_{\Theta_{-i}^{n}}\int_{0}^{1}\zeta^{j}\left(\theta,\vartheta\right)d\vartheta d\mathbf{F}\left(\theta_{-i}\right)>0\\ k & \text{if } \int_{\Theta_{-i}^{n}}\int_{0}^{1}\zeta^{j}\left(\theta,\vartheta\right)d\vartheta d\mathbf{F}\left(\theta_{-i}\right)=0\end{cases}\\
t_{i}\left(\theta_{i}\right) &= \mathrm{E}_{-i}\tau\left(\theta\right)=\int_{\Theta_{-i}^{n}}\tau\left(\theta\right)d\mathbf{F}\left(\theta_{-i}\right).
\end{aligned}$$

This is a mechanism of the form in (B1), and we will call it G. Use of the law of iterated expectations on $\rho^{j}\left(\theta\right)$ and $t_{i}\left(\theta_{i}\right)$ shows that (BB) is unaffected when switching from $\mathcal{G}$ to G. It remains to show that the surplus is unchanged, and that (IC) and (IR) continue to hold under G. The utility of agent i of type $\theta_{i}\in\Theta$ who announces $\hat{\theta}_{i}\in\Theta$ is

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\zeta^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)\omega_{i}^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)\theta_{i}-\tau\left(\hat{\theta}_{i},\theta_{-i}\right)\right]\quad\text{in mechanism }\mathcal{G} \tag{B3}$$

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\rho^{j}\left(\hat{\theta}_{i},\theta_{-i}\right)\eta_{i}^{j}\left(\hat{\theta}_{i}\right)\theta_{i}-t_{i}\left(\hat{\theta}_{i}\right)\right]\quad\text{in mechanism }G. \tag{B4}$$

If $\int_{\Theta_{-i}^{n}}\int_{0}^{1}\zeta^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)d\vartheta d\mathbf{F}\left(\theta_{-i}\right)=0$, we trivially have that the payoffs in (B3) and (B4) are identical, whereas if $\int_{\Theta_{-i}^{n}}\int_{0}^{1}\zeta^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)d\vartheta d\mathbf{F}\left(\theta_{-i}\right)>0$, we have that

$$\begin{aligned}
\mathrm{E}_{-i}\rho^{j}\left(\hat{\theta}_{i},\theta_{-i}\right)\eta_{i}^{j}\left(\hat{\theta}_{i}\right)\theta_{i} &= \mathrm{E}_{-i}\zeta^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)\frac{\mathrm{E}_{-i}\omega_{i}^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)\zeta^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)}{\mathrm{E}_{-i}\zeta^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)} && \text{(B5)}\\
&= \mathrm{E}_{-i}\omega_{i}^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)\zeta^{j}\left(\hat{\theta}_{i},\theta_{-i},\vartheta\right)\theta_{i}.
\end{aligned}$$

Trivially, $\mathrm{E}_{-i}t_{i}\left(\theta_{i}\right)=t_{i}\left(\theta_{i}\right)=\mathrm{E}_{-i}\tau\left(\theta\right)$, which combined with (B5) implies that the payoffs in (B3) and (B4) are identical. Since the equality between (B3) and (B4) were established for any i, θ_{i} and $\hat{\theta}_{i}$, it follows that all incentive and participation constraints (IC) and (IR) hold for mechanism G given that they are satisfied in mechanism $\mathcal{G}$. Moreover, [again by (B5)]

$$\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\rho^{j}\left(\theta\right)\eta_{i}^{j}\left(\theta_{i}\right)\theta_{i}\right]=\mathrm{E}_{-i}\left[\sum_{j\in\mathcal{J}}\omega_{i}^{j}\left(\theta,\vartheta\right)\zeta^{j}\left(\theta,\vartheta\right)\theta_{i}\right], \tag{B6}$$

so it follows by integration over Θ and summation over i that

$$\mathrm{E}\left[\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\rho^{j}\left(\theta\right)\eta_{i}^{j}\left(\theta_{i}\right)\theta_{i}\right]=\mathrm{E}\left[\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\zeta^{j}\left(\theta,\vartheta\right)\omega_{i}^{j}\left(\theta,\vartheta\right)\theta_{i}\right], \tag{B7}$$

By construction, we also have that $\rho^j(\theta) = \mathrm{E}_{\Xi}\zeta^j(\theta,\vartheta)$ for every θ. Thus $\mathrm{E}\left[\rho^j(\theta)\, C^j(n)\right] = \mathrm{E}\left[\zeta^j(\theta,\vartheta)\, C^j(n)\right]$, implying that

$$\sum_{j\in\mathcal{J}} \mathrm{E}\rho^j(\theta)\left[\sum_{i\in\mathcal{I}} \eta_i^j(\theta_i)\,\theta_i - C^j(n)\right] = \sum_{j\in\mathcal{J}} \mathrm{E}\zeta^j(\theta,\vartheta)\left[\sum_{i\in\mathcal{I}} \omega_i^j(\theta,\vartheta)\,\theta_i - C^j(n)\right]. \tag{B8}$$

Hence, $\mathcal{G}$ and G generate the same social surplus.

Claim B2 *For every incentive feasible mechanism of the form (B1), there exists an anonymous simple incentive feasible mechanism g of the form (5) that generates the same surplus.*

To see this, consider an incentive feasible simple mechanism G on form (B1). For $k \in \{1, ..., n!\}$, let $P_k : \mathcal{I} \to \mathcal{I}$ denote the k-th permutation of the set of agents $\mathcal{I}$. Note that $P_k^{-1}(i)$ gives the index of the agent who takes agent i's position in permutation P_k. Moreover, for any given $\theta \in \Theta^n$, let $\theta^{P_K} = \left(\theta_{P_k^{-1}(1)}, ..., \theta_{P_k^{-1}(n)}\right) \in \Theta^n$ denote the corresponding k-th permutation of θ. For each $k \in \{1, ..., n!\}$, let $G_k = \left(\left(\rho_k^j, \eta_{k1}^j, ..., \eta_{kn}^j\right)_{j=1,2}, t_{k1}, ..., t_{kn}\right)$ be given by

$$\begin{aligned}
\rho_k^j(\theta) &= \rho^j\left(\theta^{P_k}\right) \quad \forall\, \theta \in \Theta^n,\ j \in \mathcal{J}, \\
\eta_{ki}^j(\theta_i) &= \eta^j_{P_k^{-1}(i)}(\theta_i) \quad \forall\, \theta_i \in \Theta,\ j \in \mathcal{J}, i \in \mathcal{I}, \\
t_{ki}(\theta_i) &= t_{P_k^{-1}(i)}(\theta_i) \quad \forall\, \theta_i \in \Theta, i \in \mathcal{I},
\end{aligned} \tag{B9}$$

and let $g = \left(\left(\widetilde{\rho}^j, \widetilde{\eta}_1^j, ..., \widetilde{\eta}_n^j\right)_{j=1,2}, \widetilde{t}_1, ..., \widetilde{t}_n\right)$ be given by

$$\begin{aligned}
&\widetilde{\rho}^j(\theta) = \tfrac{1}{n!}\textstyle\sum_{k=1}^{n!} \rho_k^j(\theta) \quad \forall\, \theta \in \Theta^n,\ j \in \mathcal{J} \\
&\widetilde{\eta}_i^j(\theta_i) = \frac{\sum_{k=1}^{n!} \mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]\eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!} \mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]} \quad \forall\, \theta_i \in \Theta, i \in \mathcal{I},\ j \in \mathcal{J} \\
&\widetilde{t}_i(\theta_i) = \tfrac{1}{n!}\textstyle\sum_{k=1}^{n!} t_{ki}(\theta_i) \quad \forall\, \theta_i \in \Theta, i \in \mathcal{I}.
\end{aligned}$$

We now note that: (1) for each $j \in \mathcal{J}$, $\widetilde{\rho}^j(\theta) = \widetilde{\rho}^j(\theta')$ if θ' is a permutation of θ. This is immediate since the sets $\left\{\rho_k^j(\theta)\right\}_{k=1}^{n!} = \left\{\rho^j(P_k(\theta))\right\}_{k=1}^{n!}$ and $\left\{\rho_k^j(\theta')\right\}_{k=1}^{n!} = \left\{\rho^j(P_k(\theta'))\right\}_{k=1}^{n!}$ are the same; (2) for $j \in \mathcal{J}$ and each pair $i, i' \in \mathcal{I}, \widetilde{\eta}_i^j(\cdot) = \widetilde{\eta}_{i'}^j(\cdot)$. That is, the inclusion rules are the same for all agents. To see this, consider agent i and i', and suppose that $\theta_i = \theta_{i'}$. We then have that $\left\{\mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]\eta_{ki}^j(\theta_i)\right\}_{k=1}^{n!}$ and $\left\{\mathrm{E}_{-i'}\left[\rho_k^j(\theta)\right]\eta_{ki'}^j(\theta_{i'})\right\}_{k=1}^{n!}$ are identical and that $\mathrm{E}_{-i}\left[\widetilde{\rho}^j(\theta)\right] = \mathrm{E}_{-i'}\left[\widetilde{\rho}^j(\theta)\right]$; and (3) for each pair $i, i' \in \mathcal{I}, \widetilde{t}_i(\cdot) = \widetilde{t}_{i'}(\cdot)$, which is obvious since the sets $\left\{t_{ki}(\theta_i)\right\}_{k=1}^{n!}$ and $\left\{t_{ki}(\theta_i')\right\}_{k=1}^{n!}$ are identical. Together, (1), (2) and (3) establishes that g is anonymous and simple.

Now we show that g is incentive feasible and generates the same expected surplus as G. First, since G and G_k are identical except for the permutation of the agents, we have, for $k = 1, ..., n!$,

$$\sum_{j\in\mathcal{J}} \mathrm{E}\left\{\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}} \eta_{ki}^j(\theta_i)\,\theta_i^j - C^j(n)\right]\right\} = \sum_{j\in\mathcal{J}} \mathrm{E}\left\{\rho^j(\theta)\left[\sum_{i\in\mathcal{I}} \eta_i^j(\theta_i)\,\theta_i^j - C^j(n)\right]\right\}. \tag{B10}$$

Hence,

$$\begin{aligned}
\sum_{j\in\mathcal{J}} \mathrm{E}\left\{\widetilde{\rho}^j(\theta)\left[\sum_{i\in\mathcal{I}} \widetilde{\eta}^j(\theta_i)\,\theta_i^j - C^j(n)\right]\right\} &= \sum_{j\in\mathcal{J}} \mathrm{E}\left\{\frac{1}{n!}\sum_{k=1}^{n!} \rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}} \frac{\sum_{k=1}^{n!} \mathrm{E}_{-i}\rho_k^j(\theta)\,\eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!} \mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j - C^j(n)\right]\right\} \\
= \sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}} \mathrm{E}_{\theta_i}\left\{\frac{1}{n!}\sum_{k=1}^{n!} \mathrm{E}_{-i}\rho_k^j(\theta)\,\eta_{ki}^j(\theta_i)\,\theta_i^j\right\} &- \mathrm{E}\left[\frac{1}{n!}\sum_{k=1}^{n!} \rho_k^j(\theta)\right] C^j(n) \\
= \frac{1}{n!}\sum_{k=1}^{n!}\sum_{j\in\mathcal{J}} \mathrm{E}\left\{\rho_k^j(\theta)\left[\sum_{i\in\mathcal{I}} \eta_{ki}^j(\theta_i)\,\theta_i^j - C^j(n)\right]\right\} &= \sum_{j\in\mathcal{J}} \mathrm{E}\left\{\rho^j(\theta)\left[\sum_{i\in\mathcal{I}} \eta_i^j(\theta_i)\,\theta_i^j - C^j(n)\right]\right\},
\end{aligned}$$

where the last equality follows from (B10). Hence the surplus generated by g is identical to that by original mechanism G. To show that g is incentive feasible we first note that $\mathrm{E}\rho_k^j(\theta) = \mathrm{E}\rho^j(\theta)$ and $\mathrm{E}\sum_{i\in\mathcal{I}} t_{ki}(\theta_i) = \mathrm{E}\sum_{i\in\mathcal{I}} t_i(\theta_i)$ for all k, since the agents' valuations are drawn from identical distributions and G_k and G only differ in the index of the agents. Thus

$$\begin{aligned}\mathrm{E}\textstyle\sum_{i\in\mathcal{I}} \widetilde{t}_i(\theta_i) - \sum_{j\in\mathcal{J}} \mathrm{E}\widetilde{\rho}^j(\theta) C^j(n) &= \mathrm{E}\textstyle\sum_{i\in\mathcal{I}} \frac{1}{n!}\sum_{k=1}^{n!} t_{ki}(\theta_i) - \sum_{j\in\mathcal{J}} \mathrm{E}\frac{1}{n!}\sum_{k=1}^{n!} \rho_k^j(\theta) C^j(n) \\ &= \mathrm{E}\textstyle\sum_{i\in\mathcal{I}} t_i(\theta_i) - \sum_{j\in\mathcal{J}} \mathrm{E}\rho^j(\theta) C^j(n)\ ,\end{aligned}$$

so g satisfies (BB) if G does. Second, (IC) holds for any permuted mechanism, that is,

$$\mathrm{E}_{-i}\textstyle\sum_{j\in\mathcal{J}} \rho_k^j(\theta)\eta_{ki}^j(\theta_i)\,\theta_i^j - t_{ki}(\theta_i) \geq \mathrm{E}_{-i}\sum_{j\in\mathcal{J}} \rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j - t_{ki}(\widehat{\theta}_i,\theta_{-i}) \tag{B11}$$

for all $i\in\mathcal{I}$, and $\theta_i,\widehat{\theta}_i\in\Theta$. Hence,

$$\begin{aligned}\mathrm{E}_{-i}\sum_{j\in\mathcal{J}} \widetilde{\rho}^j(\theta)\widetilde{\eta}^j(\theta_i)\,\theta_i^j - \widetilde{t}(\theta_i) &= \mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\left[\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\theta)\right]\frac{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]\eta_{ki}^j(\theta_i)}{\sum_{k=1}^{n!}\mathrm{E}_{-i}\left[\rho_k^j(\theta)\right]}\theta_i^j - \frac{1}{n!}\sum_{k=1}^{n!} t_{ki}(\theta_i) \\ = \frac{1}{n!}\sum_{k=1}^{n!}\left[\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\theta)\,\eta_{ki}^j(\theta_i)\,\theta_i^j - t_{ki}(\theta_i)\right] &\geq \frac{1}{n!}\sum_{k=1}^{n!}\left[\mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j - t_{ki}(\widehat{\theta}_i,\theta_{-i})\right] \\ = \mathrm{E}_{-i}\sum_{j\in\mathcal{J}}\frac{1}{n!}\sum_{k=1}^{n!}\rho_k^j(\widehat{\theta}_i,\theta_{-i})\eta_{ki}^j(\widehat{\theta}_i,\theta_{-i})\theta_i^j - \frac{1}{n!}\sum_{k=1}^{n!} t_{ki}(\widehat{\theta}_i,\theta_{-i}) &= \sum_{j\in\mathcal{J}}\mathrm{E}_{-i}\widetilde{\rho}^j(\widehat{\theta}_i,\theta_{-i})\widetilde{\eta}_i^j\left(\widehat{\theta}_i\right)\theta_i^j - \widetilde{t}(\widehat{\theta}_i),\end{aligned} \tag{B12}$$

where the inequality follows from (B11). Hence g is satisfies (IC). Finally, g also satisfies the (IR) because (see the second line in (B12)) all the permuted mechanisms satisfy participation constraints. Proposition 1 follows by combining Claims B1 and B2. ■

PROOF OF PROPOSITION 2: This proof requires us to be explicit about the coordinates of the vector θ when permuting $\mathcal{J}$. We therefore need some extra notation for this proof (only).

Notation B1 *We will, with some abuse of notation, write* $\mathbf{F}(\theta) \equiv \Pi_{i\in\mathcal{I}} F(\theta_i)$ *and* $\mathbf{F}(\theta_{-i}) \equiv \Pi_{k\in\mathcal{I}\setminus i} F(\theta_k)$ *as the joint distribution of* θ *and* θ_{-i} *respectively. We write* $\theta_i^{-j} = \left(\theta_i^1, ..., \theta_i^{j-1}, \theta_i^{j+1}, ..., \theta_i^M\right)$ *for a type vector where good* j *has been removed. Analogously,* $\theta^{-j} = \left(\theta_1^{-j}, ..., \theta_n^{-j}\right)$ *stands for the type profile with good* j *coordinate removed for all agents and* $\theta^j = \left(\theta_1^j, ..., \theta_n^j\right)$ *is the vector collecting the valuations for good* j *for all agents. Furthermore,* $\theta_{-i}^{-j} = \left(\theta_1^{-j}, ..., \theta_{i-1}^{-j}, \theta_{i+1}^{-j}, ..., \theta_n^{-j}\right)$ *and* $\theta_{-i}^j = \left(\theta_1^j, ..., \theta_{i-1}^j, \theta_{i+1}^j ..., \theta_n^j\right)$ *are used for the vectors obtained respectively from* θ^{-j} *and* θ^j *by removing agent* i*. These conventions are used also on the distributions, so, for example,* $\mathbf{F}_{-i}^{-j}$ *denotes the cumulative distribution of* θ_{-i}^{-j}*. Conditional distributions are denoted in the natural way: for example* $\mathbf{F}_{-i}^{-j}\left(\cdot|\,\theta_i^j\right)$ *denotes the joint distribution of* θ_{-i}^{-j} *conditional on* θ_i^j*. Since no integrals are taken over subsets of the range of integration, we also conserve space and write* $\int_\theta h(\theta)\,d\mathbf{F}(\theta)$ *rather than* $\int_{\theta\in\Theta^n} h(\theta)\,d\mathbf{F}(\theta)$ *when integrating a function* h *over* θ *and similarly for integrals over various components of* θ*.*

Consider a simple anonymous incentive feasible mechanism g. For $k\in\{1, ..., M!\}$ let $P_k : \mathcal{J}\to\mathcal{J}$ be the k-th permutation of $\mathcal{J}$ and $\theta_i^{P_k} = \left(\theta_i^{P_k^{-1}(1)}, ..., \theta_i^{P_k^{-1}(M)}\right) \in \Theta$ be the permutation of θ_i when the goods are permuted according to P_k. Let $\theta^{P_k} = \left(\theta_1^{P_k}, ..., \theta_n^{P_k}\right) \in \Theta^n$ denote the corresponding permutation of θ.[1] For each $k\in\{1, ..., M!\}$ define mechanism $g_k = (\left\{\rho_k^j\right\}_{j\in\mathcal{J}}, \left\{\eta_k^j\right\}_{j\in\mathcal{J}}, t_k)$, where for every $\theta\in\Theta^n$;

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[1] To illustrate, suppose $n = 2, M = 3$, and $\theta = (\theta_1, \theta_2) = ((1,2,0),(3,2,1))$. Consider, for example, purmutation k given by $P_k(1) = 2, P_k(2) = 1, P_k(3) = 3$. Then $P_k^{-1}(1) = 2, P_k^{-1}(2) = 1, P_k^{-1}(3) = 3$ and $\theta_1^{P_k} = \left(\theta_1^{P_k^{-1}(1)}, \theta_1^{P_k^{-1}(2)}, \theta_1^{P_k^{-1}(3)}\right) = (2,1,0), \theta_2^{P_k} = \left(\theta_2^{P_k^{-1}(1)}, \theta_2^{P_k^{-1}(2)}, \theta_2^{P_k^{-1}(3)}\right) = (2,3,1), \theta^{P_k} = \left(\theta_1^{P_k}, \theta_2^{P_k}\right) = ((2,1,0),(2,3,1))$.

1. $\rho_k^j(\theta) = \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)$ for every $j \in \mathcal{J}$;[2]
2. $\eta_k^j(\theta_i) = \eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)$ for every $j \in \mathcal{J}$;[3]
3. $t_k(\theta_i) = t\left(\theta_i^{P_k}\right)$.

By construction, each g_k is simple. Each g_k is also anonymous by the anonymity of g. Using the definition of g_k and manipulating the result by observing that the labeling of the variables is irrelevant, we get:[4]

$$\begin{aligned}
\mathrm{E}\rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j &= \int_\theta \rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j d\mathbf{F}(\theta) \text{ /def of } g_k\text{/ } = \int_{\theta\in\Theta^n} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\theta_i^j d\mathbf{F}(\theta) \\
&= \int_{\theta^j}\left[\int_{\theta^{-j}} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\theta_i^j d\mathbf{F}^{-j}\left(\theta^{-j}\middle|\,\theta^j\right)\right] d\mathbf{F}^j\left(\theta^j\right) \qquad \text{(B13)} \\
\text{/relabel/} &= \int_{\theta^{P_k^{-1}(j)}}\left[\int_{\left(\theta^{-j}\right)^{P_k}} \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\middle|\,\underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}}\right)\right] d\mathbf{F}^j\left(\theta^{P_k^{-1}(j)}\right)
\end{aligned}$$

where we recall,

$$\left(\theta^{-j}\right)^{P_k} \equiv \left(\theta^{P_k^{-1}(1)}, ..., \theta^{P_k^{-1}(j-1)}, \theta^{P_k^{-1}(j+1)}, ..., \theta^{P_k^{-1}(n)}\right). \qquad \text{(B14)}$$

By exchangeability, we have

$$\begin{aligned}
& d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\middle|\,\underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th (vector) argument}}\right) \qquad \text{(B15)} \\
= \;& d\mathbf{F}^{-j}\left(\theta^{P_k^{-1}(1)}, ..., \theta^{P_k^{-1}(j-1)}, \theta^{P_k^{-1}(j+1)}, ..., \theta^{P_k^{-1}(n)} | j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)}\right) \\
= \;& d\mathbf{F}^{-j}\left(\theta^{-j} | j\text{-th (vector) argument} = \theta^{P_k^{-1}(j)}\right) \\
= \;& d\mathbf{F}^{-P_k^{-1}(j)}\left(\theta^{-P_k^{-1}(j)} | P_k^{-1}(j)\text{-th (vector) argument } = \theta^{P_k^{-1}(j)}\right);
\end{aligned}$$

and

$$d\mathbf{F}^j\left(\theta^{P_k^{-1}(j)}\right) = d\mathbf{F}^{\theta^{P_k^{-1}(j)}}\left(\theta^{P_k^{-1}(j)}\right). \qquad \text{(B16)}$$

Using (B13), (B15) and (B16), we have that

$$\begin{aligned}
& \mathrm{E}\rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j \qquad \text{(B17)} \\
= \;& \int_{\theta^{P_k^{-1}(j)}}\left[\int_{\left(\theta^{-j}\right)^{P_k}} \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-j}\left(\left(\theta^{-j}\right)^{P_k}\middle|\,\underbrace{\theta^{P_k^{-1}(j)}}_{j\text{-th argument}}\right)\right] d\mathbf{F}^j\left(\theta^{P_k^{-1}(j)}\right) \\
= \;& \int_{\theta^{P_k^{-1}(j)}}\left[\int_{\left(\theta^{-j}\right)^{P_k}} \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}^{-P_k^{-1}(j)}\left(\theta^{-P_k^{-1}(j)}\middle|\,\underbrace{\theta^{P_k^{-1}(j)}}_{P_k^{-1}(j)\text{-th argument}}\right)\right] d\mathbf{F}^{P_k^{-1}(j)}\left(\theta^{P_k^{-1}(j)}\right) \\
= \;& \int_\theta \rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\,\theta_i^{P_k^{-1}(j)} d\mathbf{F}(\theta) = \mathrm{E}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\theta_i^{P_k^{-1}(j)}.
\end{aligned}$$

Moreover, exchangeability implies that $\mathrm{E}t_k(\theta_i) = \mathrm{E}t\left(\theta_i^{P_k}\right) = \mathrm{E}t(\theta_i)$. The ex ante utility,

$$\begin{aligned}
\mathrm{E}\left[\sum_{j=1}^M \rho_k^j(\theta)\,\eta_k^j(\theta_i)\theta_i^j - t_k(\theta_i)\right] &= \left[\sum_{j=1}^M \mathrm{E}\rho^{P_k^{-1}(j)}(\theta)\,\eta^{P_k^{-1}(j)}(\theta_i)\theta_i^{P_k^{-1}(j)}\right] - \mathrm{E}t(\theta_i) \\
&= \left[\sum_{j=1}^M \mathrm{E}\rho^j(\theta)\,\eta^j(\theta_i)\theta_i^j\right] - \mathrm{E}t(\theta_i), \qquad \text{(B18)}
\end{aligned}$$

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[2] This implies that $\rho_k^{P_k^{-1}(j)}\left(\theta^{P_k}\right) = \rho^j(\theta)$ for every $j \in \mathcal{J}$.

[3] This implies that $\eta_k^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right) = \eta^j(\theta_i)$ for every $j \in \mathcal{J}$.

[4] It is important to point out that, in reaching the fourth equality in (B13), we can relabel the integrating varibles (since they are dummies) but not the integrating functions.

is thus unchanged when changing from g to g_k. The same steps as in (B13) through (B17) (only somewhat simpler) establishes that $\mathrm{E}\rho_k^j(\theta) = \mathrm{E}\rho^{P_k^{-1}(j)}$ for every j, implying that

$$\begin{aligned} \mathrm{E}\left[\ \sum_{j=1}^M\ \rho_k^j(\theta)\, C^j(n) -\ \sum_i\ t_k(\theta_i)\right] &= \left[C(n)\,\mathrm{E}\ \sum_{j=1}^M\ \rho_k^j(\theta) -\ \sum_i\ \mathrm{E}t_k(\theta_i)\right] \\ = \left[C(n)\,\mathrm{E}\ \sum_{j=1}^M\ \rho^j(\theta) -\ \sum_i\ \mathrm{E}t(\theta_i)\right] &= \mathrm{E}\left[\ \sum_{j=1}^M\ \rho^j(\theta)\, C(n) -\ \sum_i\ t(\theta_i)\right], \end{aligned} \tag{B19}$$

so the feasibility constraint is unaffected when changing from g to g_k. Next, write Write $U(\theta_i, \theta_i'; g)$ and $U(\theta_i, \theta_i'; g_k)$ for the expected utility from announcing θ_i' when the true type is θ_i in mechanisms g and g_k respectively. Next, by a calculation in the same spirit as (B13) through (B17):

$$\begin{aligned} \mathrm{E}_{-i}\rho_k^j(\theta_{-i}, \theta_i') &= \int_{\theta_{-i}} \rho_k^j(\theta_{-i}, \theta_i')\, d\mathbf{F}_{-i}(\theta_{-i})\ /\text{def of } g_k/ = \int_{\theta_{-i}} \rho^{P_k^{-1}(j)}\left(\left(\theta_{-i}, \theta_i'\right)^{P_k}\right) d\mathbf{F}_{-i}(\theta_{-i}) \\ &= \int_{\theta_{-i}^j}\left[\int_{\theta_{-i}^{-j}} \rho^{P_k^{-1}(j)}\left(\left(\theta_{-i}, \theta_i'\right)^{P_k}\right) d\mathbf{F}_{-i}^{-j}\left(\theta_{-i}^{-j}\middle|\,\theta_{-i}^j\right)\right] d\mathbf{F}_{-i}^j\left(\theta_{-i}^j\right) \\ /\text{relabel}/ &= \int_{\theta_{-i}^{P_k^{-1}(j)}}\left[\int_{\theta_{-i}^{-P_k^{-1}(j)}} \rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) d\mathbf{F}_{-i}^{-j}\left(\left(\theta_{-i}^{-j}\right)^{P_k}\middle|\,\theta_{-i}^{P_k^{-1}(j)}\right)\right] d\mathbf{F}_{-i}^j\left(\theta_{-i}^{P_k^{-1}(j)}\right) \\ /\text{exchangeability}/ &= \int_{\theta_{-i}^{P_k^{-1}(j)}}\left[\int_{\theta_{-i}^{-P_k^{-1}(j)}} \rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) d\mathbf{F}_{-i}^{-P_k^{-1}(j)}\left(\theta_{-i}^{-P_k^{-1}(j)}\middle|\,\theta_{-i}^{P_k^{-1}(j)}\right)\right] d\mathbf{F}_{-i}^j\left(\theta_{-i}^{P_k^{-1}(j)}\right) \\ &= \int_{\theta_{-i}} \rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) d\mathbf{F}_{-i}(\theta_{-i}) = \mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) \end{aligned} \tag{B20}$$

That is, the perceived probability of getting j when announcing θ_i' in mechanism g_k is the same as the perceived probability of getting good $P_k^{-1}(j)$ when announcing $\left(\theta_i'\right)^{P_k}$, so that

$$\begin{aligned} U(\theta_i, \theta_i'; g_k) &= \mathrm{E}_{-i}\ \sum_{j=1}^M\ \rho_k^j(\theta_{-i}, \theta_i')\, \eta_k^j(\theta_i')\theta_i^j - t_k(\theta_i') \\ &= \sum_{j=1}^M\ \eta_k^{P_k^{-1}(j)}((\theta_i')^{P_k})\theta_i^j \mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) - t\left(\left(\theta_i'\right)^{P_k}\right), \end{aligned} \tag{B21}$$

whereas

$$\begin{aligned} U(\theta_i, \theta_i'; g) &= \sum_{j=1}^M \eta_k^j(\theta_i')\theta_i^j \mathrm{E}_{-i}\rho_k^j(\theta_{-i}, \theta_i') - t(\theta_i') \Rightarrow \\ &\quad U(\theta_i, \theta_i'; g)\Big|_{\substack{\theta_i = \theta_i^{P_k} \\ \theta_i' = \theta_i'^{P_k}}} = \sum_{j=1}^M \eta_k^j((\theta_i')^{P_k})\theta_i^{P_k^{-1}(j)} \mathrm{E}_{-i}\rho_k^j\left(\theta_{-i}, \theta_i'^{P_k}\right) - t\left(\left(\theta_i'\right)^{P_k}\right) \\ &\quad = \sum_{j=1}^M\ \eta_k^{P_k^{-1}(j)}((\theta_i')^{P_k})\theta_i^j \mathrm{E}_{-i}\rho_k^{P_k^{-1}(j)}\left(\theta_{-i}, \theta_i'^{P_k}\right) - t\left(\left(\theta_i'\right)^{P_k}\right)\ = U(\theta_i, \theta_i'; g_k), \end{aligned} \tag{B22}$$

which establishes that type θ_i who announces θ_i' in mechanism g_k gets the same utility as type $\theta_i^{P_k}$ who announces $\left(\theta_i'\right)^{P_k}$ in mechanism g. Hence incentive compatibility and individual rationality of g_k follows from incentive compatibility and individual rationality of g. Now, construct a new mechanism $\widetilde{g} = (\left\{\widetilde{\rho}^j\right\}_{j\in\mathcal{J}}, \left\{\widetilde{\eta}^j\right\}_{j\in\mathcal{J}}, \widetilde{t})$ by letting

$$\begin{aligned} \widetilde{\rho}^j(\theta) &= \tfrac{1}{M!}\sum_{k=1}^{M!} \rho_k^j(\theta) = \tfrac{1}{M!}\sum_{k=1}^{M!} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right) \\ \widetilde{\eta}^j(\theta_i) &= \frac{\sum_{k=1}^{M!} \eta_k^j(\theta_i)\,\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!} \mathrm{E}_{-i}\rho_k^j(\theta)} = \frac{\sum_{k=1}^{M!} \eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right) \mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}{\sum_{k=1}^{M!} \mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)} \\ \widetilde{t}(\theta_i) &= \tfrac{1}{M!}t_k(\theta_i) = \tfrac{1}{M!}t\left(\theta_i^{P_k}\right) \end{aligned} \tag{B23}$$

let $P : \mathcal{J} \rightarrow \mathcal{J}$ be an arbitrary perturbation of the set of goods. Then,

$$\widetilde{\rho}^{P^{-1}(j)}\left(\theta^P\right) = \tfrac{1}{M!}\sum_{k=1}^{M!} \rho^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right) = \tfrac{1}{M!}\sum_{k=1}^{M!} \rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right) = \widetilde{\rho}^j(\theta)\ , \tag{B24}$$

since the sets $\left\{\rho^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)\right\}_{k=1}^{M!}$ and $\left\{\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)\right\}_{k=1}^{M!}$ are identical. Furthermore

$$\begin{aligned}\widetilde{\eta}^{P^{-1}(j)}\left(\theta_i^P\right) &= \frac{\sum_{k=1}^{M!}\eta_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta_i^P\right)^{P_k}\right)\mathrm{E}_{-i}\rho_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^{P_k^{-1}\left(P^{-1}(j)\right)}\left(\left(\theta^P\right)^{P_k}\right)} \\ &= \frac{\sum_{k=1}^{M!}\eta^{P_k^{-1}(j)}\left(\theta_i^{P_k}\right)\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho^{P_k^{-1}(j)}\left(\theta^{P_k}\right)} = \widetilde{\eta}^j\left(\theta_i\right)\end{aligned} \tag{B25}$$

for the same reason. It is obvious that $\widetilde{t}\left(\theta_i^P\right) = \widetilde{t}\left(\theta_i\right)$, which together with (B24) and (B25) establishes that $\widetilde{g}$ is symmetric. To complete the proof we need to show that $\widetilde{g}$ is incentive feasible and generates the same surplus as g. We note that

$$\begin{aligned}&\mathrm{E}\widetilde{\rho}^j\left(\theta\right)\widetilde{\eta}^j\left(\theta_i\right)\theta_i^j = \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\mathrm{E}\rho_k^j\left(\theta\right)\frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i)\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j \\ &= \tfrac{1}{M!}\mathrm{E}_{\theta_i}\textstyle\sum_{k=1}^{M!}\left[\mathrm{E}_{-i}\rho_k^j\left(\theta\right)\frac{\sum_{k=1}^{M!}\eta_k^j(\theta_i)\mathrm{E}_{-i}\rho_k^j(\theta)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j(\theta)}\theta_i^j\right] = \tfrac{1}{M!}\mathrm{E}\left[\textstyle\sum_{k=1}^{M!}\eta_k^j\left(\theta_i\right)\rho_k^j\left(\theta\right)\theta_i^j\right] \\ &\Rightarrow \mathrm{E}\textstyle\sum_{j=1}^{M}\left[\widetilde{\rho}^j\left(\theta\right)\widetilde{\eta}^j\left(\theta_i\right)\theta_i^j - \widetilde{t}\left(\theta_i\right)\right] = \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\mathrm{E}\left[\textstyle\sum_{j=1}^{M}\eta_k^j\left(\theta_i\right)\rho_k^j\left(\theta\right)\theta_i^j - t_k\left(\theta_i\right)\right] \\ &= \mathrm{E}\left[\textstyle\sum_{j=1}^{M}\eta^j\left(\theta_i\right)\rho^j\left(\theta\right)\theta_i^j - t\left(\theta_i\right)\right],\end{aligned}$$

where the last equality follows from (B17) and (B18), which establishes that the ex ante utility from $\widetilde{g}$ and g are the same for all agents. Moreover,

$$\begin{aligned}&\mathrm{E}\left[\textstyle\sum_{j=1}^{M}\widetilde{\rho}^j\left(\theta\right)C^j\left(n\right) - \sum_{i=1}^{n}\widetilde{t}\left(\theta_i\right)\right] = \mathrm{E}\left[C\left(n\right)\textstyle\sum_{j=1}^{M}\tfrac{1}{M!}\sum_{k=1}^{M!}\rho_k^j\left(\theta\right) - \sum_{i=1}^{n}\sum_{k=1}^{M!}\tfrac{1}{M!}t_k\left(\theta_i\right)\right] \\ &= \textstyle\sum_{k=1}^{M!}\mathrm{E}\left[C\left(n\right)\textstyle\sum_{j=1}^{M}\rho_k^j\left(\theta\right) - \sum_{i=1}^{n}t_k\left(\theta_i\right)\right] \\ &= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\mathrm{E}\left[\textstyle\sum_{j=1}^{M}\rho^j\left(\theta\right)C\left(n\right) - \sum_{i=1}^{n}t\left(\theta_i\right)\right] = \mathrm{E}\left[\textstyle\sum_{j=1}^{M}\rho^j\left(\theta\right)C^j\left(n\right) - \sum_{i=1}^{n}t\left(\theta_i\right)\right],\end{aligned}$$

where the third equality follows from (B19). So the budget balance constraint is unaffected. All incentive compatibility constraints hold since,

$$\begin{aligned}U(\theta_i,\theta_i';\widetilde{g}) &= \sum_{j=1}^{M}\widetilde{\eta}^j(\theta_i')\theta_i^j\mathrm{E}_{-i}\widetilde{\rho}^j\left(\theta_{-i},\theta_i'\right) - \widetilde{t}\left(\theta_i'\right) \\ &= \frac{\sum_{k=1}^{M!}\eta_k^j\left(\theta_i'\right)\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'\right)}{\sum_{k=1}^{M!}\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'\right)}\mathrm{E}_{-i}\left[\tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\rho_k^j\left(\theta_{-i},\theta_i'\right)\right] - \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}t_k\left(\theta_i'\right) \\ &= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}\left[\eta_k^j\left(\theta_i'\right)\mathrm{E}_{-i}\rho_k^j\left(\theta_{-i},\theta_i'\right) - t_k\left(\theta_i'\right)\right] \\ &= \tfrac{1}{M!}\textstyle\sum_{k=1}^{M!}U(\theta_i,\theta_i';g_k) \ \leq \frac{1}{M!}\sum_{k=1}^{M!}U\left(\theta;g_k\right) = U(\theta;\widetilde{g}).\end{aligned}$$

where the third equality follows from (B21). By the same calculation, $U(\theta;\widetilde{g}) = \frac{1}{M!}\sum_{k=1}^{M!}U(\theta;g_k) \geq 0$, since all participation constraints hold for each k. This completes the proof. ■

PROOF OF LEMMA 1: The only variables that are not automatically in a compact set are the transfers. However, $t_i\left(\mathbf{l}\right) \leq Ml < Mh$ and $t_i\left(\theta_i\right) - t_i\left(\theta_i|l_k\right) \leq m\left(\theta_i\right)h + \left[M - m\left(\theta_i\right)\right]l < Mh$. Recursive application of (15) therefore implies that we may bound $t_i\left(\theta_i\right)$ from above by M^2h. Since this is also an upper bound for the difference between $t_i\left(\theta_i\right)$ and $t_i\left(\theta_i|l_k\right)$, it follows from (11) that we may bound $t_i\left(\theta_i\right)$ from below by $-M^2h$. Hence, existence of a solution to (14) follows from Weierstrass maximum theorem. ■

PROOF OF LEMMA 2: Let $m\left(\theta_i\right) = m\left(\widehat{\theta}_i\right) = m$, let $\lambda_i\left(\theta_i,\theta_i'\right)$ denote the multiplier associated with one of the $m-1$ downwards adjacent constraints for type θ_i and let $\lambda_i\left(\widehat{\theta}_i,\widehat{\theta}_i'\right)$ denote the multiplier associated with one of the $m-1$ downwards adjacent constraints for type $\widehat{\theta}_i$. Proposition 2 ensures that provision and inclusion rules are symmetric and by use of strong duality in linear programming we find that it is without loss to assume that $\lambda_i\left(\theta_i,\theta_i'\right) = \lambda_i\left(\widehat{\theta}_i,\widehat{\theta}_i'\right)$. ■

PROOF OF LEMMA 3: Fix m and let $\theta_i \in \Theta$ with $m(\theta_i) = m\{1, ..., M-1\}$. Consider the incentive constraints that involves θ_i :

$$0 \leq \underbrace{\sum_{\theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^{M} \rho^j(\theta)\, \eta_i^j(\theta_i)\, \theta_i^j - t_i(\theta_i)}_{\text{Term } A} - \underbrace{\sum_{\theta_{-i}} \beta_{-i}(\theta_{-i}) \sum_{j=1}^{M} \rho^j(\theta_{-i}, \theta_i | l_k)\, \eta_i^j(\theta_i | l_k)\, \theta_i^j - t_i(\theta_i | l_k)}_{\text{Term } B} \tag{B26}$$

There are m different adjacent downward deviations from θ_i (i.e. to replace a single high-value coordinate in θ_i with a low-value); thus $t_i(\theta_i)$ enters in Term A for m conditions. From Lemma 2, the multiplier associated with each of these conditions is $\lambda(m)$. Moreover, there are $M-m$ types of θ_i' with $m\left(\theta_i'\right) = m+1$ such that an adjacent downward deviation from θ_i' can "turn into" type θ_i; thus $t_i(\theta_i)$ enters in Term B for $M-m$ downwards incentive constraints for types with $m+1$ high valuations. Again from Lemma 2, the multiplier associated with each of these conditions is $\lambda(m+1)$. Finally, $t_i(\theta_i)$ appears in the balanced budget constraint (11). The first order condition with respect to $t_i(\theta_i)$ can thus be written as

$$\begin{aligned} & -m\lambda(m) + (M-m)\lambda(m+1) + \Lambda\beta_i(\theta_i) \\ = \quad & -m\lambda(m) + (M-m)\lambda(m+1) + \Lambda\frac{m!\,(M-m)!}{M!}\beta_m = 0, \end{aligned} \tag{B27}$$

where we used (7) for the first equality.

When $m(\theta_i) = 0$, i.e. when $\theta_i = \mathbf{l} = (\mathbf{l}, ..., \mathbf{l})$, there is no adjacent downward deviation from θ_i; instead, $t_i(\mathbf{1})$ appears in the individual rationality constraint (10) for type-$\mathbf{l}$. Thus the optimality condition with respect to $t_i(\mathbf{l})$ is:

$$-\lambda(0) + \lambda(1)M + \Lambda\beta_i(\mathbf{l}) = 0.$$

Similarly, the optimality condition with respect to $t_i(\mathbf{h})$ where $\mathbf{h} = (h, ..., h)$ is

$$-M\lambda(M) + \Lambda\beta_i(\mathbf{h}) = 0.$$

For $m = M$, condition (B27) reads $M\lambda(M) = \Lambda\beta_M$, implying that Lemma 3 is also true for $m = M$. Now suppose that $\lambda(m)$ is given by the expression in (16) for some $m \leq M$. The optimality condition (B27) with respect to $t_i\left(\theta_i'\right)$ where $m\left(\theta_i'\right) = m-1$ then reads:

$$\begin{aligned} 0 &= -(m-1)\lambda(m-1) + (M-m+1)\lambda(m) + \frac{(m-1)!\,(M-m+1)!}{M!}\beta_{m-1} \\ &= -(m-1)\lambda(m-1) + \frac{(M-m+1)}{m}m\lambda(m) + \frac{(m-1)!\,(M-m+1)!}{M!}\beta_{m-1} \\ &= -(m-1)\lambda(m-1) + \frac{(M-m+1)}{m}\frac{m!\,(M-m)!}{M!}\Lambda\sum_{j=m}^{M}\beta_j + \frac{(m-1)!\,(M-m+1)!}{M!}\beta_{m-1} \\ &= -(m-1)\lambda(m-1) + \frac{(m-1)!\,(M-m+1)!}{M!}\Lambda\sum_{j=m-1}^{M}\beta_j, \end{aligned}$$

where the third equality follows from induction hypothesis. Thus, (16) holds also for $m-1$. The result follows from induction. ■

PROOF OF LEMMA 7: **[Part 1]** Consider types θ_i and θ_i' with $m(\theta_i) = m$ and $m\left(\theta_i'\right) = m+1$. For ease of notation, define $U(m, m)$ and $U(m+1, m+1)$ as the payoff of truth-telling for type m and $m+1$; and denote the payoff from a type with $m+1$ high valuations to announce a type with m high valuations as $U(m+1, m)$, and the payoff from a type with m high valuations to announce $m+1$ high valuations as $U(m, m+1)$. $U(m+1, m)$ and $U(m, m+1)$ are respectively given as:

$$\begin{aligned} U(m+1, m) &= \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i})\left\{P^h(\theta_{-i}, m)\, mh + P^l(\theta_{-i}, m)\,\eta(m)\left[(M-m-1)\,l + h\right]\right\} - t(m), \\ U(m, m+1) &= \sum_{\theta_{-i} \in \Theta_{-i}} \beta_{-i}(\theta_{-i})\left[P^h(\theta_{-i}, m+1)(mh + l) + P^l(\theta_{-i}, m+1)\,\eta(m+1)(M-m-1)\,l\right] - t(m+1). \end{aligned}$$

We then have that

$$
\begin{aligned}
& U(m,m) - U(m,m+1) \\
= & \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) P^h(\theta_{-i},m)\, mh + P^l(\theta_{-i},m)\, \eta(m)\, (M-m)\, l - t(m) \\
& - \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m+1)\, [mh+l] + P^l(\theta_{-i},m+1)\, \eta(m+1)\, (M-m-1)\, l\right] - t(m+1) \\
= & \underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) P^h(\theta_{-i},m)\, mh + P^l(\theta_{-i},m)\, \eta(m)\, \{(M-m-1)\, l + h\} - t(m)}_{=U(m+1,m)} \\
& + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) P^l(\theta_{-i},m)\, \eta(m)\, (l-h) \\
& \underbrace{- \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m+1)\, (m+1)\, h + P^l(\theta_{-i},m+1)\, \eta(m+1)\, (M-m-1)\, l\right] - t(m+1)}_{=U(m+1,m+1)} \\
& + \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m+1)\, [h-l]\right] \\
= & \underbrace{U(m+1,m) - U(m+1,m+1)}_{=0 \text{ by hypothesis}} + \underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left\{P^h(\theta_{-i},m+1) - P^l(\theta_{-i},m)\, \eta(m)\right\} [h-l]}_{\geq 0 \text{ as } P^h(\theta_{-i},m+1) - P^l(\theta_{-i},m)\eta(m) \geq 0} \\
\geq & \; 0.
\end{aligned}
$$

[Part 2] [Downward Incentive Constraints] The proof is by induction. Pick an arbitrary m. Assume that there is some $K < m$ such that a type with m high valuations has no incentives to pretend to be of any type with $k \in \{m-1, ..., K\}$ high valuations. It follows that

$$
\begin{aligned}
U(m,m) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},m)\, mh + P^l(\theta_{-i},m)\, \eta(m)\, (M-m)\, l\right] - t(m) \\
&\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},k)\, Kh + P^l(\theta_{-i},K)\, \eta(K)\, (M-m)\, l + (m-K)\, h\right] - t(K) \\
&= U(m,K)
\end{aligned}
\tag{B28}
$$

is satisfied by hypothesis. By assumption the downwards adjacent incentive constraint for type K holds, implying that

$$
\begin{aligned}
U(K,K) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K)\, Kh + P^l(\theta_{-i},K)\, \eta(K)\, (M-K)\, l\right] - t(K) \\
&\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i}) \left[P^h(\theta_{-i},K-1)\, (K-1)\, h + P^l(\theta_{-i},K-1)\, \eta(K-1)\, (M-K)\, l + h\right] - t(K-1) \\
&= U(K,K-1)
\end{aligned}
\tag{B29}
$$

But, the payoff of announcing type $K-1$ for type m is

$$\begin{aligned}
&\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left[P^{h}\left(\theta_{-i},K-1\right)\left(K-1\right)h+P^{l}\left(\theta_{-i},K\right)\eta\left(K-1\right)\left(M-m\right)l+\left(m-K+1\right)h\right]-t\left(K-1\right)\\
&\quad=\underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left[P^{h}\left(\theta_{-i},K-1\right)\left(K-1\right)h+P^{l}\left(\theta_{-i},K\right)\eta\left(K-1\right)\left(M-K\right)l+h\right]-t\left(K-1\right)}_{\text{RHS in (B29)}}\\
&\qquad\qquad+\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{l}\left(\theta_{-i},K-1\right)\eta\left(K-1\right)\left(m-K\right)\left(h-l\right)\\
&\quad /(\text{B29})/\leq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left[P^{h}\left(\theta_{-i},K\right)Kh+P^{l}\left(\theta_{-i},K\right)\eta\left(K\right)\left(M-K\right)l\right]-t\left(K\right)\\
&\qquad\qquad+\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{l}\left(\theta_{-i},K-1\right)\eta\left(K-1\right)\left(m-K\right)\left(h-l\right)\\
&\quad=\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left[P^{h}\left(\theta_{-i},K\right)Kh+P^{l}\left(\theta_{-i},K\right)\eta\left(K\right)\left(M-m\right)l+\left(m-K\right)h\right]-t\left(K\right)\\
&\qquad\qquad-\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{l}\left(\theta_{-i},K\right)\eta\left(K\right)\left(m-K\right)\left(h-l\right)\\
&\qquad\qquad+\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{l}\left(\theta_{-i},K-1\right)\eta\left(K-1\right)\left(m-K\right)\left(h-l\right)\\
&\Bigg/\begin{array}{c}(\text{B28})\text{ and}\\ P^{l}(\theta_{-i},K-1)\eta(K-1)\\ \leq P^{l}(\theta_{-i},K)\eta(K)\end{array}\Bigg/\leq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left[P^{h}\left(\theta_{-i},m\right)mh+P^{l}\left(\theta_{-i},m\right)\eta\left(m\right)\left(M-m\right)l\right]-t\left(m\right),
\end{aligned}$$

implying that m has no incentive to mimic $K-1$. By induction it follows that all downwards constraints are satisfied. ■

[Upward Incentive Constraints] The proof is by induction. Let $K>m$ and assume that

$$\begin{aligned}
U\left(m,m\right) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left\{P^{h}\left(\theta_{-i},m\right)mh+P^{l}\left(\theta_{-i},m\right)\eta\left(m\right)\left(M-m\right)l\right\}-t\left(m\right)\\
&\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left\{P^{h}\left(\theta_{-i},k\right)\left[mh+\left(k-m\right)l\right]+P^{l}\left(\theta_{-i},k\right)\eta\left(k\right)\left(M-k\right)l\right\}-t\left(k\right)\\
&= U\left(m,k\right)
\end{aligned}$$

for all $k\in\{m+1,...,K\}$. If $K=M$, all upwards constraint hold by assumption. If $K<M$, the upwards adjacent constraint for type K implies that

$$\begin{aligned}
U\left(K,K\right) &= \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{h}\left(\theta_{-i},K\right)Kh+P^{l}\left(\theta_{-i},K\right)\eta\left(K\right)\left(M-K\right)l-t\left(K\right) \qquad \text{(B30)}\\
&\geq \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{h}\left(\theta_{-i},K+1\right)\left[Kh+l\right]+P^{l}\left(\theta_{-i},K+1\right)\eta\left(K+1\right)\left(M-K-1\right)l-t\left(K+1\right)\\
&= U\left(K,K+1\right).
\end{aligned}$$

We then note that

$$\begin{aligned}
& U\left(m,m\right)-U\left(m,K+1\right) \;\geq\; U\left(m,K\right)-U\left(m,K+1\right) \\
= \; & \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left\{P^{h}\left(\theta_{-i},K\right)\left[mh+\left(K-m\right)l\right]+P^{l}\left(\theta_{-i},K\right)\eta\left(K\right)\left(M-K\right)l\right\}-t\left(K\right) \\
& -\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{h}\left(\theta_{-i},K+1\right)\left[mh+\left(K+1-m\right)l\right]+P^{l}\left(\theta_{-i},K+1\right)\eta\left(K+1\right)\left(M-K-1\right)l-t\left(K+1\right) \\
= \; & \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left\{P^{h}\left(\theta_{-i},K\right)Kh+P^{l}\left(\theta_{-i},K\right)\eta\left(K\right)\left(M-K\right)l\right\}-t\left(K\right) \\
& +\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left\{P^{h}\left(\theta_{-i},K\right)\left(K-m\right)\left(l-h\right)\right\} \\
& -\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{h}\left(\theta_{-i},K+1\right)\left[Kh+l\right]+P^{l}\left(\theta_{-i},K+1\right)\eta\left(K+1\right)\left(M-K-1\right)l-t\left(K+1\right) \\
& +\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)P^{h}\left(\theta_{-i},K+1\right)\left(K-m\right)\left(h-l\right) \\
= \; & \underbrace{U\left(K,K\right)-U\left(K,K+1\right)}_{\geq 0 \text{ by (B30)}}+\underbrace{\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left[P^{h}\left(\theta_{-i},K+1\right)-P^{h}\left(\theta_{-i},K\right)\right]\left(K-m\right)\left(h-l\right)}_{\geq 0 \text{ by monotonicity of } P^{h}},
\end{aligned}$$

implying that type m has no incentive to mis-report as type $K+1$. ■

[Part 3] The solution to (14) violates an incentive constraint in (8) if there exists θ_i, θ_i' such that

$$\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\sum_{j=1}^{M}\rho^{j}\left(\theta\right)\eta_i^j\left(\theta_i\right)\theta_i^j-t_i\left(\theta_i\right)<\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\sum_{j=1}^{M}\rho^{j}\left(\theta_{-i},\theta_i'\right)\eta_i^j\left(\theta_i'\right)\theta_i^j-t_i\left(\theta_i'\right). \tag{B31}$$

Since θ_i is exchangeable and costs are identical for all goods, we can apply Proposition 2 (in conjunction with Lemma 4) to conclude that it is without loss of generality to assume that there exists $\left\{\eta\left(m\right), P^{h}\left(\theta_{-i},m\right), P^{l}\left(\theta_{-i},m\right), t\left(m\right)\right\}_{m=0}^{M}$, such that:

- $\eta_i^j\left(\theta_i\right)=\eta\left(m\right)$ for every $\left(\theta_i,j\right)$ such that $\theta_i^j=l$ for good j and $m\left(\theta_i\right)=m$;
- $\rho^j\left(\theta_{-i},\theta_i'\right)=P^{h}\left(\theta_{-i},m\right)$ for every $\left(\theta_i,j\right)$ such that $\theta_i^j=h$ for good j and $m\left(\theta_i\right)=k$;
- $\rho^j\left(\theta_{-i},\theta_i'\right)=P^{l}\left(\theta_{-i},m\right)$ for every $\left(\theta_i,j\right)$ such that $\theta_i^j=l$ for good j and $m\left(\theta_i\right)=k$;
- $t_i\left(\theta_i\right)=t\left(m\right)$ for every θ_i such that $\theta_i^k=h$ and $m\left(\theta_i\right)=k$.

Consider an arbitrary announcement θ_i' with $m\left(\theta_i'\right)=m'$. Let $r\leq\min\left\{m',m\right\}$ be the number of coordinates such that $\theta_i^j=\theta_i'^j=h$; and let $s\leq\min\left\{M-m',M-m\right\}$ be the number of coordinates such that $\theta_i^j=\theta_i'^j=l$. We can then express the failure of an incentive constraint for the full problem in (B31) as

$$\begin{aligned}
& \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left[P^{h}\left(\theta_{-i},m\right)mh+P^{l}\left(\theta_{-i},m\right)\eta\left(m\right)\left(M-m\right)l\right]-t\left(m\right) \\
< \; & \sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}\left(\theta_{-i}\right)\left\{P^{h}\left(\theta_{-i},m'\right)\left[rh+\left(m'-r\right)l\right]+P^{l}\left(\theta_{-i},m'\right)\eta\left(m'\right)\left[sl+\left(M-m'-s\right)h\right]\right\}-t\left(m'\right).
\end{aligned} \tag{B32}$$

We note that if $r<m'$ and $s<M-m'$, then it is possible to announce a type θ_i'' (that differs from θ_i') with $m\left(\theta_i''\right)=m\left(\theta_i'\right)=m'$, but there are $r+1$ coordinates with $\theta_i'^j=\theta_i''^j=h$ and $s+1$ coordinates with $\theta_i'^j=\theta_i''^j=l$.

The utility for agent i with type θ_i' from announcing θ_i'' is, using the mechanism described above,

$$\begin{aligned}
&\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\left\{P^h(\theta_{-i},m')\left[(r+1)h+(m'-r-1)l\right]+P^l(\theta_{-i},m')\eta(m')\left[(s+1)l+(M-m'-s-1)h\right]\right\}-t(m') \\
= &\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\left\{P^h(\theta_{-i},m')\left[rh+(m'-r)l\right]+P^l(\theta_{-i},m')\eta(m')\left[sl+(M-m'-s)h\right]\right\}-t(m') \\
&+\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\left[P^h(\theta_{-i},m')-P^l(\theta_{-i},m')\eta(m')\right](h-l) \\
\geq &\sum_{\theta_{-i}\in\Theta_{-i}} \beta_{-i}(\theta_{-i})\left\{P^h(\theta_{-i},m')\left[rh+(m'-r)l\right]+P^l(\theta_{-i},m')\eta(m')\left[sl+(M-m'-s)h\right]\right\}-t(m')
\end{aligned}$$

where the inequality follows from the assumed monotonicity [i.e. $P^h(\theta_{-i},m')\geq P^l(\theta_{-i},m')$] and $\eta(m')\leq 1$. This is a violation of the upwards incentive constraints, a contradiction to the postulate of the Lemma that all upwards incentive constraints hold. Thus, we conclude that a failure of an incentive constraint implies that either $r=m<m'$ and $s=M-m'$, in which case an upwards incentive constraint fails or $r=m'<m$ and $s=M-m$ in which case a downwards incentive constraint fails. ■

PROOF OF LEMMA 9: To prove this lemma, it turns out to be useful to consider a related auxiliary problem which aims to maximize the average probability of provision (instead of social welfare) under the same constraints as the relaxed problem (14):

$$\max_{\{\rho,\eta,t\}} \sum_{\theta\in\Theta^n}\beta(\theta)\sum_{j=1}^{M}\rho^j(\theta) \qquad \text{(B33)}$$
$$\text{s.t. (15), (10), (11).}$$

We first show in Claim B3 below that the characterization of the solution to (B33) is qualitatively similar to the constrained welfare problem (14):

Claim B3 *Let (ρ,η,t) be an optimal solution to (B33) and assume that $\sum_{\theta_{-i}\in\Theta_{-i}}\beta_{-i}(\theta_{-i})\rho^j(\theta_i,\theta_{-i})>0$. Then,*

1. *conditional on provision, all consumers get access to all their high valuation goods and the inclusion rule for low valuation goods is given by:*

$$\eta_i^j(\theta_i)=\eta(m)\equiv\begin{cases}0 & \text{if } G_m(1)<0\\ z\in[0,1] & \text{if } G_m(1)=0\\ 1 & \text{if } G_m(1)>0;\end{cases} \qquad \text{(B34)}$$

2. *there exists some $\Lambda\geq 0$ such that the provision rule for good j satisfies*

$$\rho^j(\theta)=\begin{cases}0 & \text{if } 1+\Lambda\sum_{m=0}^{M}\left[H^j(\theta,m)h+L^j(\theta,m)\frac{\max\{0,G_m(1)\}}{\beta_m(M-m)}-cn\right]<0\\ z\in[0,1] & \text{if } 1+\Lambda\sum_{m=0}^{M}\left[H^j(\theta,m)h+L^j(\theta,m)\frac{\max\{0,G_m(1)\}}{\beta_m(M-m)}-cn\right]=0\\ 1 & \text{if } 1+\Lambda\sum_{m=0}^{M}\left[H^j(\theta,m)h+L^j(\theta,m)\frac{\max\{0,G_m(1)\}}{\beta_m(M-m)}-cn\right]>0\end{cases} \qquad \text{(B35)}$$

Proof of Claim B3: The derivation of the inclusion rules follow the analysis of the constrained welfare problem step by step and is omitted.

To derive the provision rule, note that the optimality conditions for $\rho^j(\theta)$ associated with the problem (B33) may be written as

$$\begin{aligned}
&\beta(\theta)+\sum_{m=0}^{M}\lambda(m)m\left[H^j(\theta,m)\beta_{-i}(\theta_{-i})h+L^j(\theta,m)\eta(m)l\right]\\
&-\sum_{m=0}^{M-1}\lambda(m+1)\left\{H^j(\theta,m)\beta_{-i}(\theta_{-i})(M-m)h+L^j(\theta,m)\beta_{-i}(\theta_{-i})\eta(m)\left[(M-m)l+(h-l)\right]\right\}\\
&-\Lambda\beta(\theta)cn+\gamma^j(\theta)-\phi^j(\theta)\ =\ 0,
\end{aligned}$$

together with the complementary slackness conditions. By noting that $\frac{\beta_{-i}(\theta_{-i})}{\beta(\theta)} = \frac{1}{\beta_i(\theta_i)}$, we can write the condition as

$$\begin{aligned}
&1+\sum_{m=0}^{M}\lambda(m)\left[H^j(\theta,m)\frac{m}{\beta_i(\theta_i)}h+L^j(\theta,m)\frac{m}{\beta_i(\theta_i)}\eta(m)\,l\right]\\
&-\sum_{m=0}^{M-1}\lambda(m+1)\left\{H^j(\theta,m)\frac{(M-m)}{\beta_i(\theta_i)}h+L^j(\theta,m)\frac{1}{\beta_i(\theta_i)}\eta(m)\left[(M-m)\,l+(h-l)\right]\right\}\\
&-\Lambda cn+\frac{\gamma^j(\theta)-\phi^j(\theta)}{\beta(\theta)} \;=\; 0.
\end{aligned}$$

Collecting terms, we get

$$\begin{aligned}
1+&\sum_{m=0}^{M}H^j(\theta,m)\left\{\frac{h}{\beta_i(\theta_i)}\left[\lambda(m)\,m-\lambda(m+1)\,(M-m)\right]\right\}\\
&+\sum_{m=0}^{M}L^j(\theta,m)\frac{\eta(m)}{\beta_i(\theta_i)}\left\{\lambda(m)\,ml-\lambda(m+1)\left[(M-m)\,l+(h-l)\right]\right\}\\
&-\Lambda cn+\frac{\gamma^j(\theta)-\phi^j(\theta)}{\beta(\theta)} \;=\; 0.
\end{aligned}$$

Using the difference equation for the multipliers in (17), we can simplify this further to:

$$1+\sum_{m=0}^{M}H^j(\theta,m)\,h\Lambda+\sum_{m=0}^{M}L^j(\theta,m)\left\{\Lambda\eta(m)\,l-\frac{\eta(m)}{\beta_i(\theta_i)}\lambda(m+1)\,(h-l)\right\}-\Lambda cn+\frac{\gamma^j(\theta)-\phi^j(\theta)}{\beta(\theta)} \;=\; 0.$$

Using (A11), we can eliminate $\lambda(m+1)$ and get

$$1+\sum_{m=0}^{M}H^j(\theta,m)\,h\Lambda+\sum_{m=0}^{M}L^j(\theta,m)\left\{\Lambda\eta(m)\,l-\frac{\eta(m)\,(h-l)}{\beta_m(M-m)}\Lambda\sum_{j=m+1}^{M}\beta_j\right\}-\Lambda cn+\frac{\gamma^j(\theta)-\phi^j(\theta)}{\beta(\theta)} \;=\; 0.$$

Furthermore,

$$\Lambda\eta(m)\,l-\frac{\eta(m)\,(h-l)}{\beta_m(M-m)}\Lambda\sum_{j=m+1}^{M}\beta_j=\frac{\Lambda\eta(m)}{\beta_m(M-m)}\left[\beta_m(M-m)\,l-(h-l)\sum_{j=m+1}^{M}\beta_j\right]=\frac{\Lambda}{\beta_m(M-m)}\max\left\{0,G_m(1)\right\},$$

where $G_m(\cdot)$ is defined in (20). Substituting this back into (A10) gives

$$1+\Lambda\sum_{m=0}^{M}\left[H^j(\theta,m)\,h+L^j(\theta,m)\frac{\max\left\{0,G_m(1)\right\}}{\beta_m(M-m)}-cn\right]+\frac{\gamma^j(\theta)-\phi^j(\theta)}{\beta(\theta)}=0.$$

which, by combining with the complementary slackness conditions, gives the result. ■

Before getting into the details of the proof, it is also useful to observe that (B35) can be rewritten as

$$\rho^j(\theta)=\begin{cases}0 & \text{if } 1+\Lambda\left[\sum_{i=1}^{n}Z^j(\theta_i)-cn\right]<0\\ z\in[0,1] & \text{if } 1+\Lambda\left[\sum_{i=1}^{n}Z^j(\theta_i)-cn\right]=0\\ 1 & \text{if } 1+\Lambda\left[\sum_{i=1}^{n}Z^j(\theta_i)-cn\right]>0,\end{cases}$$

where

$$Z^j\left(\theta^i\right)=\begin{cases}h & \text{if } \theta_i^j=h\\ \frac{\max\{0,G_m(1)\}}{\beta_m(M-m)} & \text{if } \theta_i^j=l \text{ and } \theta_i^k=h \text{ for exactly } m \text{ goods } k\in\mathcal{J}.\end{cases}$$

The point with this formulation is that $\left\{Z^j\left(\theta^i\right)\right\}_{i=1}^{n}$ is a sequence of i.i.d. random variables. Observe that $Z^j\left(\theta^i\right)$ is bounded below by 0 and above by h, so the variance is bounded. This allows us to use the central limit theorem to establish a "generalized paradox of voting", which simply states the intuitively obvious fact that the influence of a single agent approaches zero as the number of agents goes out of bounds. To state this, we will now need to be careful about the fact that the solution depends on the number of agents in the economy, and a mechanism with n agents will therefore now be denoted by (ρ_n, η_n, t_n) and the multiplier on the resource constraint will be denoted by Λ_n.

(Proof of Lemma 9, continued): Consider a mechanism that solves (B33) first. As $Z^j\left(\theta^i\right) \in [0, h]$ for every $\theta^i \in \Theta$, it follows that for any pair $\left(\theta_i', \theta_i''\right)$ we have

$$\begin{aligned} \mathrm{E}\left[\rho_n^j(\theta)\,|\theta_i'\right] - \mathrm{E}\left[\rho_n^j(\theta)\,|\theta_i''\right] &\leq \Pr\left[1 + \Lambda_n\left(h + \sum_{k\neq i} Z^j(\theta_k)\right) - cn \geq 0\right] - \Pr\left[1 + \Lambda_n\left(0 + \sum_{k\neq i} Z^j(\theta_k)\right) - cn > 0\right] \\ &= \Pr\left[\frac{1-cn}{\Lambda_n} - h \leq \sum_{k\neq i} Z^j(\theta_k) \leq \frac{1-cn}{\Lambda_n}\right]. \end{aligned}$$

Let $\sigma = \mathrm{VAR}\left(Z^j\left(\theta^i\right)\right)$. We can then rewrite the probability statement as

$$\Pr\left[k_n \leq \frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}} \leq k_n + \frac{h}{\sigma\sqrt{n-1}}\right],$$

where $k_n \equiv \frac{1-cn}{\Lambda_n\sigma\sqrt{n-1}} - \frac{h}{\sigma\sqrt{n-1}} - \frac{\mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}}$. As σ is finite, $\mathrm{E}\left\{Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)\right\} = 0$, and $\left\{Z^j\left(\theta^i\right)\right\}_{i=1}^n$ is an i.i.d. sequence, we know that the central limit theorem is applicable, thus $\frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}}$ is asymptotically distributed as a standard normal distribution. Moreover, $\frac{h}{\sigma\sqrt{n-1}} \to 0$, which together with the convergence in distribution implies that for every real number k and any $\varepsilon > 0$, there exists $N < \infty$ such that

$$\Pr\left[k \leq \frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}} \leq k + \frac{h}{\sigma\sqrt{n-1}}\right] \leq \frac{1}{\sqrt{2\pi}}\int_k^{k+\varepsilon} \exp\left(\frac{y^2}{2}\right) dy + \varepsilon$$

for $n \geq N$. But, the standard normal is symmetric and single-peaked, so for $n \geq N$,

$$\begin{aligned} \Pr\left[k_n \leq \frac{\sum_{k\neq i} Z^j(\theta_k) - \mathrm{E}Z^j(\theta_i)}{\sigma\sqrt{n-1}} \leq k_n + \frac{h}{\sigma\sqrt{n-1}}\right] &\leq \frac{1}{\sqrt{2\pi}}\int_{k_n}^{k_n+\varepsilon} \exp\left(\frac{y^2}{2}\right) dy + \varepsilon \\ &\leq \frac{1}{\sqrt{2\pi}}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \exp\left(\frac{y^2}{2}\right) dy + \varepsilon \to 0 \end{aligned}$$

as $\varepsilon \to 0$. The result follows.

For the case in which $\{\rho_n, \eta_n, t_n\}_{n=1}^{\infty}$ is a sequence from the solution to (14), we replace $Z^j(\theta_i)$ above with

$$Z_n^j(\theta_i) = \begin{cases} h & \text{if } \theta_i^j = h \\ \frac{\max\{0, G_m(\Phi_n)\}}{\beta_m(M-m)} & \text{if } \theta_i^j = l \text{ and if } \theta_i^k = h \text{ and } m(\theta_i) = m. \end{cases}$$

By choice of subsequences such that $G_m(\Phi_n) \to G_m^*$, we can approximate $\frac{\max\{0, G_m(\Phi_n)\}}{\beta_m(M-m)}$ with $\frac{\max\{0, G_m^*\}}{\beta_m(M-m)}$. The rest of the argument follows the one above step by step. ■