Optimum Growth in an Aggregative Model of Capital Accumulation[1]

1. This paper elaborates the problem of optimum saving first discussed by Frank Ramsey in 1928 [4]. For this purpose a centralized, closed economy is postulated; it is assumed to be adequately described by the aggregative model first closely analyzed by Solow [5].[2] The social welfare to be maximized is then represented by the total of the discounted utility of consumption per capita, where a general concave utility index is employed. Within this framework it is demonstrated that there is a unique optimum growth path, and the qualitative nature of this path as well as its relation to the golden rule growth path are discussed.

2. The basic premises of our model can be described as follows: A single homogeneous output, $Y(t)$, is produced with the use of two homogeneous factors, labor, $L(t)$, and capital goods, $K(t)$, under the direction of a central planning board. The technically efficient possibilities for production, which are unchanging over time, are known to the planning board and are summarized in an aggregate production function. This relation exhibits constant returns to scale, positive marginal productivities, and a diminishing marginal rate of substitution. In addition, it is known that roundaboutness in production is extremely productive when capital is relatively very scarce, while capital saturation only occurs when capital is relatively very abundant. If we denote by lower case letters quantities measured in terms of labor, these assumptions about production can be represented by

$$y(t) = f(k(t)), \tag{1}$$

with

$$f(k) > 0,\ f'(k) > 0,\ f''(k) < 0 \text{ for } k > 0, \tag{2}$$

and

$$\lim_{k\to 0} f'(k) = \infty,\ \lim_{k\to\infty} f'(k) = 0. \tag{3}$$

The labor force and population both grow exogenously at the positive rate n. Hence, quantities measured in terms of the labor force are equivalent, but for a scale factor γ, to quantities per capita. As the central planning board has the authority to require all able persons to work, by assumption (2) the whole labor force will always be productively employed.

Current output constitutes the only source of new goods to this closed economy, and it can either be used to satisfy current consumption requirements, in which case it is instantaneously consumed, or added to the capital stock, in which case it depreciates at the fixed positive rate μ. Letting $c(t)$ be the rate of current consumption per capita, and

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After the original version of this paper was completed, a very similar analysis by Koopmans [1] came to our attention. We draw on his results in discussing the limiting case, where the effective social discount rate goes to zero.

[2] The results to be presented carry over directly into the two-sector model introduced by Uzawa [7], in which the available techniques for producing investment goods differ from those for consumption goods. As it simplifies the analysis, and also because no further insight is gained otherwise, we adopt an aggregative approach.

$z(t)$ the current rate of gross investment per capita, the allocation alternatives available at time t yield the relations

(4) $$c(t) + z(t) = y(t),\ c(t) \geqq 0,\ z(t) \geqq 0,$$

and

(5) $$\dot{k}(t) = z(t) - \lambda k(t), \text{ with } \lambda = n + \mu > 0.$$

At the present time $t = 0$ the central planning board can choose any *feasible* growth path $(c(t), z(t), k(t) : t \geqq 0)$ which starts from the historically given initial capital-labor ratio $0 < k(0) < \infty$ and satisfies the conditions (1), (4) and (5). It desires to choose the *optimum* feasible growth path with respect to the criterion of maximizing social welfare.

3. The central planning authority's concept of social welfare is related to the ability of the economy to provide consumption goods over time. In particular, welfare at any point of time is measured by a utility index of current consumption per capita $U(c(t))$ weighted by the current population $\frac{1}{\gamma}L(t)$. This index exhibits

(6) $$U'(c) > 0,\ U''(c) < 0 \text{ for all } c > 0,$$

positive but diminishing marginal utility (or alternatively, a diminishing marginal rate of substitution between any two generations), as well as

(7) $$\lim_{c \to 0} U'(c) = \infty,$$

one differentiable generalization of the necessity of avoiding extremely low levels of consumption per capita. We emphasize that (7) entails that an optimum path will never specify a zero level of consumption per capita.

The utility index is time invariant. However, along with population growth, the central planning authority recognizes that consumption tomorrow is not the same thing as consumption today. For this reason, it takes the politically pragmatic view that its planning obligation is stronger to present and near future generations than to far removed future generations. This view is implemented in practice by discounting future welfare at a positive rate greater than the population growth rate, $\rho > n$, rather than by short term planning—as the central planning authority also recognizes that, for any finite planning period, terminal capital simply represents the feasible growth paths and hence potential welfare beyond the horizon.

Finally, then, the social welfare associated with any particular feasible growth path is given by the functional representing total welfare

(8) $$\int_0^\infty U(c(t)) \frac{L(t)}{\gamma} e^{-\rho t} dt = \frac{1}{\gamma}\int_0^\infty U(c(t)) e^{-\delta t} dt, \text{ with } \delta = \rho - n > 0.$$

(Note that hereafter dependence on the variable t is not explicitly denoted unless necessary for clarity.)

4. The problem confronting the technical staff of the central planning board is thus to specify a particular feasible growth path which maximizes (8). In order to characterize such a path, we appeal to the general formulation of the classical calculus of variations developed by Pontryagin and co-workers ([4], especially theorem 7, p. 69, and the discussions on pp. 189-191, 298-300). By introducing the imputed price of a unit of gross investment per head $q = q(t)$, which entails an imputed value of gross national product per capita

(9) $$\psi = U(c) + qz,$$

and then applying the Maximum Principle to the Hamiltonian expression representing the present imputed value of net national product per capita, the following theorem is obtained:

In addition to feasibility (1), (4) *and* (5), *the necessary conditions for an optimum growth path are that there exists a continuous imputed price such that*

$$(10)\qquad \dot{q} = \frac{d\left(\int_t^\infty \frac{\partial\psi}{\partial k} e^{-(\delta+\lambda)(\tau-t)} d\tau\right)}{dt} = (\delta + \lambda)q - U'(c)f'(k),$$

and

$$(11)\qquad \lim_{t\to\infty} qe^{-\delta t} = 0,$$

the imputed price changes as if, say, the central planning board exercises perfect foresight with respect to the marginal, imputed value product of capital, while the present imputed price goes to zero as the date of valuation recedes indefinitely; and

$$(12)\qquad \left(\frac{\partial\psi}{\partial z}\right)_{c\,=\,f(k)\,-\,z} = -U'(c) + q \leqq 0, \text{ with equality for } z > 0,$$

given the capital-labor ratio and the imputed price, current allocation maximizes the imputed value of gross national product at each point of time.

It should be clear that all the above valuations are in terms of the utility index of the rate of per capita consumption.

By virtue of the assumptions of a diminishing marginal rate of substitution (2) and diminishing marginal utility (6), the conditions just enumerated are also sufficient, and characterize a unique optimum growth path (if such exists). We merely sketch the proof of these assertions: Suppose we have found a feasible growth path (c^0, z^0, k^0) and an imputed price q^0 which satisfy (10), (11) and (12). Then, for any other feasible growth path (c^1, z^1, k^1), it follows that [1]

$$\int_0^\infty \{U(c^0) - U(c^1)\}e^{-\delta t}dt = \int_0^\infty \{[U(c^0) - U(c^1)] + U'(c^0).$$

$$[(f(k^0) - c^0 - z^0) - (f(k^1) - c^1 - z^1)] + q^0[(z^0 - \lambda k^0 - \dot{k}^0) - (z^1 - \lambda k^1 - \dot{k}^1)]\}e^{-\delta t}dt$$

$$= \int_0^\infty \{[U(c^0) - U(c^1) - U'(c^0)(c^0 - c^1)] + [(q^0 - U'(c^0))(z^0 - z^1)]$$

$$+ [(\dot{q}^0 - q^0(\delta + \lambda) + U'(c^0)f'(k^0))(k^0 - k^1)]$$

$$+ U'(c^0).\ [f(k^0) - f(k^1) - f'(k^0)(k^0 - k^1)]\}e^{-\delta t}dt - [q^0e^{-\delta t}(k^0 - k^1)]_0^\infty \geqq 0,$$

with strict inequality if $k^1(\tau) \neq k^0(\tau)$ at some point $\tau > 0$. Note that this proof (as well as a meaningful formulation of our problem) requires the fact that the capital-labor ratio in our stylized economy is bounded from above. This follows from the observation that even if all output is invested the capital-labour ratio falls whenever $\bar{k} < k$, where under assumptions (2) and (3) $\bar{k} < \infty$ is uniquely defined from (5) by setting $\dot{k} = 0$ with $z = f(k)$

$$(13)\qquad f(\bar{k}) = \lambda\bar{k}.$$

Thus, on any feasible growth path it must be true that

$$k \leqq \max(k(0), \bar{k}) < \infty.$$

[1] Here and in the sequel we are implicitly assuming that the allocation process (4) on any feasible growth path is mildly well-behaved, e.g., that c and z are piecewise continuous on any finite interval.

5. We are now in a position to describe, for the benefit of the central planning board, the unique optimum growth path. It is instructive to begin by ignoring the historically given initial capital-labor ratio, and to ask whether there is any path which satisfies all the other optimality conditions. Clearly, one such path is the non-trival singular solution to the pair of differential equations (5) and (10) given the allocation conditions (4) and (12), denoted by (c^*, z^*, k^*) and q^*, and uniquely defined by

(14) $$\dot{q}^* = 0 \text{ or } f'(k^*) = \delta + \lambda,$$

(15) $$\dot{k}^* = 0 \text{ or } z^* = \lambda k^*,$$

(16) $$c^* = f(k^*) - z^*,$$

and

(17) $$q^* = U'(c^*),$$

from assumptions (2), (3), and (6). (c^*, z^*, k^*) thus represents the *quasi-stationary* (*optimum*) *path*, i.e., the balanced path corresponding to the one initial capital-labor ratio $k(o) = k^*$ which would be voluntarily maintained forever as optimum.

Notice that the quasi-stationary path is independent of the form of the utility index, but rather (given the underlying structure of the economy) depends only on the effective social discount rate δ. This enables us to relate it to a familiar result in the literature dealing with the choice between balanced growth paths: namely, if we let $\delta \to 0$, then (14) - (17) become

(14′) $$f'(\tilde{k}) = \lambda,$$

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(17′) $$\tilde{q} = U'(\tilde{c}),$$

which simply states that the limiting quasi-stationary path is nothing more than the golden rule path, denoted by $(\tilde{c}, \tilde{z}, \tilde{k})$, discussed, for example, by Phelps [2]. We shall have reason to refer to this result when discussing the general limiting optimum path.

But the quasi-stationary path is more than just a curiosum to a central planning board intent on moving along an optimum path from the historically given initial capital-labor ratio. For, expanding (5) and (10) around the point (k^*, q^*), it is easily shown that the characteristic roots of the resulting linear system are given by

$$\tfrac{1}{2}\left(\delta \pm \sqrt{\delta^2 + 4\frac{f''(k^*)q^*}{U''(c^*)}}\right),$$

which, from (2) and (6), are real and opposite in sign. Hence, the quasi-stationary path is a saddle point, suggesting in light of the transversality condition (11) that its stable branches are logical candidates for the unique optimum path given $k(o) \neq k^*$. This conjecture is easily verified by examining the behavior of the pair of differential equations (5) and (10) in the positive quadrant of the (k, q) plane:

First, consider the behavior of the capital-labor ratio k. As $\dot{k} = 0$ if and only if $z = \lambda k > 0$, (4) and (12) yield

(18) $$q = U'(f(k) - \lambda k)$$

for the curve above (below) which $\dot{k} > 0$ $(\dot{k} < 0)$. Noticing further that

(19) $$\left(\frac{dq}{dk}\right)_{\dot{k}=0} = U''(f(k) - \lambda k)(f'(k) - \lambda) \gtreqless 0 \text{ as } k \gtreqless \tilde{k},$$

and that

(20) $$\left(\lim_{k \to 0} q\right)_{\dot{k}=0} = \left(\lim_{k \to \bar{k}} q\right)_{\dot{k}=0} = \infty,$$

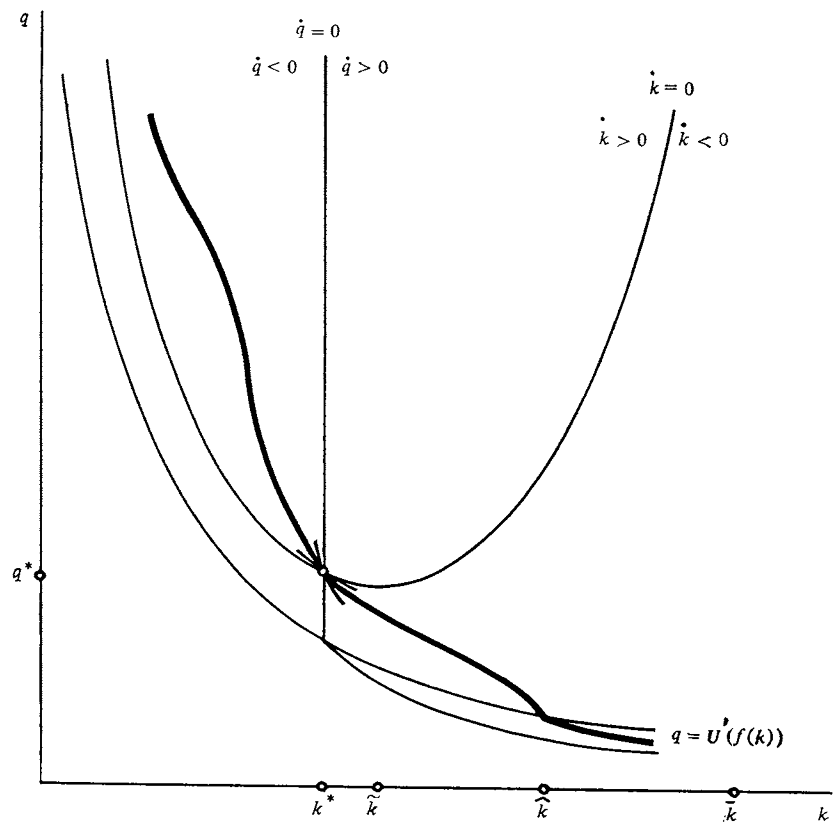

FIGURE I
The Optimum Growth Path

the capital-labor ratio behaves as depicted in Figure I.

Second, consider the behavior of the imputed price q. Here, (4), (10) and (12) yield

(21) $$\dot{q} = (\delta + \lambda - f'(k))q, \text{ for } z > 0$$
$$= (\delta + \lambda)q - U'(f(k))f'(k), \text{ for } z = 0,$$

from which it follows that $\dot{q} > 0$ (<0) to the right (left) of the curve defined by

(22) $$k = k^*, \text{ for } q > U'(f(k^*))$$
$$q = \frac{U'(f(k))f'(k)}{\delta + \lambda}, \text{ for } q \leqq U'(f(k^*)).$$

Moreover, from (22)

$$\left(\frac{dq}{dk}\right)_{\substack{\dot{q} = 0 \\ q \leq U'(f(k^*))}} = \frac{U''(f'(k))f'(k)^2 + U'(f'(k))f''(k)}{\delta + \lambda} < 0$$

so that the imputed price behaves as depicted in Figure I.

Finally, for completeness, we describe the curve above (below) which $z > 0$ ($=0$)

(23) $$q = U'(f(k)),$$

with

$$\frac{dq}{dk} = U''(f'(k))f'(k) < 0.$$

(23) lies below (18) everywhere, intersects (22) uniquely, and is thus as also illustrated in Figure I.

Collecting the results of the three preceding paragraphs, we see that the stable branches of the saddle point (k^*, q^*) are as depicted by the heavy arrows in Figure I. Notice, in particular, that on these paths (a) $k(q)$ is strictly increasing (decreasing) for $k(o) < k^*$, but strictly decreasing (increasing) for $k(o) > k^*$, and (b) there may exist some capital-labor ratio $k^* < \hat{k} < \infty$ such that specialization in consumption occurs when $\hat{k} \leqq k$. It also easily follows from (4) and (12) that c behaves as k on these paths.

From the foregoing, we can summarize the central result in a theorem:

For any initial capital-labor ratio $k(o)$ an initial imputed price of investment goods $q(o)$ can be chosen in such a way that the path starting from these values and satisfying the optimality conditions asymptotically approaches the quasi-stationary path (c^*, z^*, k^*). This path is the unique optimum path. Moreover, on this unique optimum path either the capital-labor ratio k and consumption per capita c are both strictly increasing, if $k(o) < k^*$, or they are both strictly decreasing, if $k(o) > k^*$.

6. It would be of some interest to determine the behavior of gross investment per capita z, or more specifically, the gross saving rate s, defined by

$$s = \frac{z}{f(k)}, \tag{24}$$

on the unique optimum path. Surprisingly, without making additional assumptions about the forms of the utility index and production function, it is impossible to state precisely how s behaves, except, of course, that

$$0 < \lim_{t \to \infty} s = \frac{\lambda k^*}{f(k^*)} < \frac{\lambda \tilde{k}}{f(\tilde{k})} < 1,$$

it approaches a definite limit. To see this, we need only differentiate (24) and rearrange the result to obtain (for $k \leqq \hat{k}$)

$$\dot{s} = \frac{1}{f(k)^2}\{cf'(k)k - \dot{c}f(k)\}. \tag{25}$$

It is clear that on the optimum path the two terms within the braces are opposite in sign, so that unless their relative values are known, the sign of $\dot{s}$ is indeterminate. This is an interesting result in itself, for it suggests that with at least some plausible forms for the functions $U(\cdot)$ and $f(\cdot)$, the optimum gross saving rate may increase (decrease) steadily or increase (decrease) and then decrease (increase) starting from $k(0) < k^*$ $(>k^*)$. That is, for example, even though a relatively capital-poor economy will pursue optimum growth by steadily increasing its capital-labor ratio, it may accomplish this by saving relatively more in the future than in the present. In this connection it is worthwhile remarking that for the Cobb-Douglas production function and a similar (precisely, a constant elasticity of marginal utility) utility function, starting from any $k(0) > 0$ the optimum gross saving rate may increase steadily, remain constant or decrease steadily, depending on the particular parameter values chosen.[1]

7. We also mention that the limiting optimum path is well defined by (1), (4), (5), (10) and (12) with $\delta = 0$.[2] It likewise coincides with the stable branches of a saddle point, namely the point $(\tilde{k}, \tilde{q})$ corresponding to the limiting quasi-stationary or golden rule path. Furthermore it is easily verified that on this limiting optimum path

$$\frac{d(\Psi - q\lambda k)}{dt} = 0, \tag{26}$$

[1] This follows somewhat indirectly by transforming the optimality conditions into a pair of differential equations in k and s, and then analyzing the behavior of this system when $f(k) = Ak^{\alpha}$ with $0 < \alpha < 1$ and $U'(c) = c^{\beta}$ with $\beta < 0$.

[2] In this section we restrict attention to cases for which $k(0) \leqq \hat{k}$ (or alternatively, like Koopmans [1], we drop the assumption that $z \geqq 0$) so that (12) is always satisfied with equality on the limiting optimum path.

the imputed value of net national product per capita is constant and thus

$$\Psi - q\lambda k = \lim_{t\to\infty} \Psi - q\lambda k = U(\tilde{c}), \tag{27}$$

equals individual welfare on the golden rule path. Hence, substituting into (27) from (5), (9) and (12) yields

$$U'(c)\dot{k} = U(\tilde{c}) - U(c), \tag{28}$$

which is nothing but the Keynes-Ramsey rule with bliss replaced by golden rule individual welfare. (28) suggests a result, originally proved rigorously by Koopmans [1], that the limiting optimum path maximizes

$$\int_0^\infty \{U(c) - U(\tilde{c})\}dt, \tag{29}$$

the amount by which welfare actually achieved exceeds golden rule welfare. Our previous analysis enables us to sketch a proof of this interesting conclusion:

Denote the limiting optimum path (c^0, z^0, k^0) with corresponding imputed price q^0, and any other feasible path (c^1, z^1, k^1). Also define

$$\varliminf_{t\to\infty} k^1 = \underline{k}^1 \leqq \varlimsup_{t\to\infty} k^1 = \bar{k}^1.$$

There are two cases to consider, (a) $\underline{k}^1 = \bar{k}^1 = \tilde{k}$ and (b) $\underline{k}^1 \neq \tilde{k}$ or $\bar{k}^1 \neq \tilde{k}$. If $\underline{k}^1 = \bar{k}^1 = \tilde{k}$, then from our earlier sufficiency proof it follows that

$$\int_0^T \{U(c^1) - U(c^0)\}dt \leqq$$

$$-\int_0^T U'(c^0)[f(k^0) - f(k^1) - f'(k^1)(k^0 - k^1)]dt + [q^0(k^0 - k^1)]_0^T,$$

and therefore that

$$\int_0^\infty \{U(c^1) - U(c^0)\}dt \leqq$$

$$-\lim_{T\to\infty} \int_0^T U'(c^0)[f(k^0) - f(k^1) - f'(k^1)(k^0 - k^1)]dt \leqq 0, \tag{30}$$

with strict inequality if $k^1(\tau) \neq k^0(\tau)$ for some $\tau > 0$. But from (12) and (28) we know that

$$\left| \int_0^\infty \{U(c^0) - U(\tilde{c})\}dt \right| = \int_0^\infty q^0 \, | \dot{k}^0 | \, dt \leqq | \tilde{k} - k(o) | \max (q^0(o), U'(\tilde{c})) < \infty. \tag{31}$$

Hence, as

$$\int_0^T \{U(c^1) - U(\tilde{c})\}dt = \int_0^T \{U(c^1) - U(c^0)\}dt + \int_0^T \{U(c^0) - U(\tilde{c})\}dt,$$

(30) and (31) together imply that

$$\int_0^\infty \{U(c^1) - U(\tilde{c})\}dt \leqq \int_0^\infty \{U(c^0) - U(\tilde{c})\}dt,$$

with strict inequality if $k^1(\tau) \neq k^0(\tau)$ for some $\tau > 0$.

On the other hand, if $\underline{k}^1 \neq \tilde{k}$ or $\bar{k}^1 \neq \tilde{k}$, then again from our earlier sufficiency proof (recalling that (14′)-(17′) is itself a limiting optimum path when $k(o) = \tilde{k}$)

$$\int_0^T \{U(c^1) - U(\tilde{c})\}dt \leqq - \int_0^T \tilde{q}[f(\tilde{k}) - f(k^1) - f'(\tilde{k})(\tilde{k} - k^1)]dt + [\tilde{q}(\tilde{k} - k^1)]_0^T$$

or

$$\overline{\lim_{T\to\infty}} \int_0^T \{U(c^1) - U(\tilde{c})\}dt \leqq \tag{32}$$
$$- \lim_{T\to\infty} \int_0^T \tilde{q}[f(\tilde{k}) - f(k^1) - f'(\tilde{k})(\tilde{k} - k^1)]dt - \underline{\lim}_{T\to\infty} [\tilde{q}(k^1(T) - k(o))].$$

The second term in the last expression is finite, while from the continuity of k on any feasible path the first term diverges to $-\infty$. Hence, for this second case, (30) and (32) together imply

$$\int_0^\infty \{U(c^1) - U(\tilde{c})\}dt < \int_0^\infty \{U(c^0) - U(\tilde{c})\}dt.$$

8. Our central results are very similar to those of Srinivasan [6] and Uzawa [8]. However, as they both assume that the utility index is simply consumption per capita, we can assert that the introduction of a diminishing marginal rate of substitution between generations' welfare has intrinsic merit. The subsidiary results—first, that even in our extremely simplified economy the behavior of the optimum gross saving rate is ambiguous, and second, that though we eschew the (somewhat artificial) foreseeable bliss level of Ramsey [4], it reappears in a different guise when we attempt to interpret the limiting optimum path—are also noteworthy. Finally, our (also somewhat artificial) positive effective social discount rate glosses over a difficult problem, the proper weighting of future generations in the concept of social welfare, in particular, when the population is growing. This appears a worthwhile area for further study.

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