Summary of Macro

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Arrow-Debreu is the basic underlying model. Sequence of markets (SME) is an extension that allows trade in each spot, with assets being the only way to transfer resources between spots. But it has loans which are never exercised in equilibrium. Since they are redundant, SMEE gets rid of loans. However, it keeps the price of loans, as it is equal to the rate of return on capital. Interest rates give macroeconomists warm fuzzy feelings, so use those instead of stupid asexual rates of return. SMEE is complicated because the optimization is a huge infinite-dimensional pain. Recursive competitive equilibrium uses the standard Bellman equation trick to reduce a huge infinite-dimensional problem into a single-dimensional one. Also, it restricts attention to stationary equilibria, which have the advantage of giving a unique answer to questions like "if wealth goes up, what happens to labor supply?"

Definition 0.1 (Commodity space). $L = (\ell^3)_\infty$

Definition 0.2 (Dual space). *If* $(X, \|\cdot\|)$ *is a normed space, then its dual* $(X^*, \|\cdot\|_\infty)$ *is*

$$X^* = \{f \in C^0(X) : f \text{ is linear}\}.$$

I will write prices as elements of L^*.

Definition 0.3 (Investment consumption set). *The investment consumption set with depreciation* δ *on capital is*

$$\begin{aligned} X(k_0) = \{x \in L : &\text{there exists } c, k \in l_\infty \text{ such that} \\ &c_t \geq 0, k_t \geq 0, x_{1t} + (1-\delta)k_t = c_t + k_{t+1}, x_{2t} \in [-k_t, 0], x_{3t} \in [-1, 0]\}. \end{aligned}$$

Definition 0.4 (Production set). $Y = \prod_{t=1}^\infty Y_t$, *where* $Y_t = \{y \in \mathbb{R}^3 : y_{1t} \leq F(-y_{2t}, -y_{3t})\}$.

TODO: when are these compact? (Under which topologies?)

TODO: this is a representative agent version. Should generalize notation.

Definition 0.5 (Arrow Debreu Equilibrium). *Suppose* $X, Y \subseteq L$. *Then* $(x^*, y^*, p^*) \in X \times Y \times L^*$ *is an ADE if*

- *Representative agent maximizes:*
$$x^* \in \underset{\{x \in X : p^*(x) = 0\}}{\arg\max} \; u(x).$$
- *Firm maximizes:*
$$y^* \in \underset{y \in Y}{\arg\max} \; p^*(y).$$
- *Market clears:* $x^* = y^*$.

Definition 0.6 (Steady state). *An Arrow-Debreu equilibrium* (x^*, y^*, p^*) *is a steady state if* $x^*_{t+1} = x^*_t$ *for all* $t \in \mathbb{N}$.

Definition 0.7 (Allocation). *An* H*-household allocation is a tuple* $(x^*, y^*) \in L^H \times L$.

Definition 0.8 (Feasible allocation). *An allocation is feasible if*

$$\sum_{h \in H} x_h^* = y^*.$$

Note that we usually assume $H = \{1\}$.

Definition 0.9 (Representative agent Pareto optimal). *An allocation $(x^*, y^*) \in L^H \times L$ is Pareto optimal if*

1. *(x^*, y^*) is feasible.*
2. *There is no $(\hat{x}, \hat{y})$ such that $u(\hat{x}_h) \geq u(x_h^*)$ for all households $h \in H$, and there is a household h' such that $u(\hat{x}_{h'}) > u(x_{h'}^*)$.*

Theorem 0.10 (First Welfare Theorem). *If u is locally non-satiated and (x^*, y^*, p^*) is a weak Arrow-Debreu equilibrium, then x^* is Pareto optimal.*

Proof. Suppose otherwise, that there is another feasible allocation $(\hat{x}, \hat{y})$ with $\hat{x} = \hat{y}$ such that $u(\hat{x}_{h'}) > u(x_{h'}^*)$.

Consider the households $H^=$ with $u(\hat{x}_h) = u(x_h^*)$. By local non-satiation, they must have $p^*\hat{x}_h = px_h^*$. (Otherwise, the x_h^* must be in the interior of the budget set, and cannot be optimal.)

Similarly, the households $H^>$ with $u(\hat{x}_h) > u(x_h^*)$ must have $p^*\hat{x}_h > p^*x_h^*$.

Summing up, $p^* \sum_{h \in H} \hat{x}_h > p^* \sum_{h \in H} x_h^*$. So $p^* \sum_{h \in H} \hat{x}_h = p^*\hat{y} > p^*y^*$, contradicting the condition that firms profit maximize. □

Theorem 0.11 (Hahn-Banach Theorem). *Let L is a normed vector space and $A, B \subseteq L$ be convex. If either*

1. *B has an interior point and $A \cap \operatorname{int}(B) = \emptyset$, or*
2. *L is finite dimensional and $A \cap B = \emptyset$*

then there is some $\phi \in L^* \backslash 0$ and some $c \in \mathbb{R}$ such that $\phi(b) \leq c \leq \phi(a)$ for all $a \in A$ and all $b \in B$.

Theorem 0.12 (Second welfare theorem). *Suppose u is continuous, increasing*[1] *and strictly-quasiconcave, X is convex, Y is convex and either finite-dimensional or has non-empty interior. If (x^*, y^*) is a Pareto optimal allocation, there exists $p^* \in L^*$ such that (x^*, y^*, p^*) is an Arrow-Debreu equilibrium with transfers.*

Proof. **Step 1 (Quasi-equilibrium)**: Let $Z = \sum_{h \in H} U(x_h^*)$. This convex by the assumption that u is strictly quasiconcave. Moreover, since u is increasing, $x^* \in \operatorname{bd} Z$.

By the Hahn-Banach theorem, there is $p^* \in L^*$ and $c \in \mathbb{R}$ such that $p^*y \leq c \leq p^*x$ for all $y \in Y$ and $x \in Z$. So, if $u(\hat{x}) \geq u(x^*)$, then $p^*\hat{x} \geq c$. Similarly, if $y \in Y$, then $p^*y \leq c$. So $p^*\hat{x} \geq p^*y$, and (x^*, y^*, p^*) is a quasi-equilibrium.

Step 2: By the duality theorem, (x^*, y^*, p^*) is an Arrow-Debreu equilibrium. □

- SBWT. FIXME: transfers? using interior crap on X to apply the duality theorems.
- Conditions for continuous linear functionals.

Second welfare theorem gives us continuous prices under fairly strong assumptions. But we need more to get a dot-product representation. (TODO)

[1] We only need this for one household. Since we're doing representative agents, this observation isn't worth much.

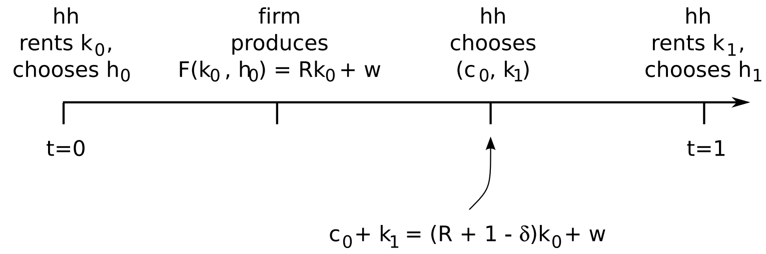

Figure 1: The growth model without profits.

Definition 0.13 (Arrow Debreu Social Planner's Problem).

$$\arg\max_{k_{t+1}\in[0,F(k_t,1)]} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $c_t = F(k_t, 1) - k_{t+1}$.

Existence: SPP has a unique solution. So Pareto set is a singleton. PO = ADE, since SBWT imples PO subset ADE and FBWT implies ADE subset PO.

TODO: talk about q^* being linear. TODO: define an SME model in which loans need to be paid back.

Definition 0.14 (Sequence of Markets Equilibrium). $(x^*, q^*, y^*, l^*, R^*)$ *is an SME if*

- $(x^*_{it}, l^*_{t+1}) \in \arg\max \sum_t \beta^t u(c_t)$ *such that* $c_t + k_{t+1} + l_{t+1} = R^*_t l_t + \hat{q}_{2t} k_t + \hat{q}_{3t}$ *for all* t*, where* $\hat{q}_{it} = q^*_{it}/q^*_{1t}$.
- $y^* \in \max_{y\in Y} q^* \cdot y$.
- $x^* = y^*$.
- $l^*_{t+1} = 0$ *for all* t.

Definition 0.15 (SME (Easy)). $(c^*_t, k^*_{t+1}, w^*_t, R^*_t)$ *is an SMEE given* k_0 *if*

1. $c^*_t, k^*_{t+1} \in \arg\max_{c_t, k_{t+1}} \sum_t \beta^t u(c_t)$ *such that* $c_t + k_{t+1} = R^*_t k_t + w^*_t$.
2. $(k^*_t, 1) \in \arg\max_{k_t, n_t} f(k_t, n_t) - R^*_t k_t - w^*_t n_t$.
3. $c_t + k_{t+1} = F(k_t, 1)$. *[Can be dropped with Walras' law.]*

Figure 1 indicates how these should be interpreted. Note that the equality $Rk_t + w_t = f(k_t, h_t)$ is only true when there are zero profits.

Definition 0.16 (Recursive Competitive Equilibrium). $(g^*, G^*, h^*, H^*, R^*, w^*)$

- *Consumer's problem:* $V(K, a; G) = \max_{c,n,a'}[u(c, n) + \beta V(K', a'; G)]$ *such that* $c + a' = w^*(K)n + R^*(K)a$ *and* $K' = G(K)$. $(g^*, h^*) = \arg\max_{a',n} \ldots$.
- *Firm's problem:*

$$(Id(K), N(K)) \in \arg\max_{(K',N')\in\mathbb{R}^2} (1, R^*(K), w^*(K)) \cdot (F(K', N'), -K', -N').$$

- *Representative agents:* $g^*(K, K; G^*) = G^*(K)$ *and* $h^*(K, K; N) = H^*(K)$*, for all* K *and* N.

- *Market clearing:* $h^*(K) = N(K)$, $F(K, N(K)) = c + a'$.

Definition 0.17 (Recursive Competitive Equilibrium (Victor)). (g^*, G^*, h^*, H^*)

- *Consumer's problem: Let* $V(K, a; G) = \max_{c,n,a'}[u(c, n) + \beta V(K', a'; G)]$ *such that* $c + a' = R(K)a + w(K)n$ *and* $K' = G(K)$ *where* $R(K) = F_N(K, N)$, $w(K) = F_K(K, N)$ *and* $N = H(K)$.

  Then $(g^*, h^*) = \arg\max_{a',n} \cdots$.

- *Representative agents:* $g^*(K, K; G^*) = G^*(K)$ *and* $h^*(K, K; N) = H^*(K)$, *for all* K *and* N.

Note that $V^*(K, a) = V(K, a; G^*)$.

Definition 0.18 (State space). *Let* $Z = \{z_1, \ldots, z_{nz}\}$ *be any finite set.*

Definition 0.19 (History set). *Let* $H_t = Z^t$ *and* $H = \cup_{t=0}^{\infty} H_t$.

Definition 0.20 (Commodity space). $L = (\mathbb{R}^3)^H$.

Definition 0.21 (Investment Consumption Set (with shocks)). *The investment consumption set with depreciation* δ *on capital is*

$$X(k_0) = \{x \in L : \text{there exists } c, k \in \mathbb{R}^H \text{ such that}$$
$$c_t(h_t) \geq 0, k_t(h_t) \geq 0, x_{1t}(h_t) + (1-\delta)k_t(h_{t-1}) = c_t(h_t) + k_{t+1}(h_t), x_{2t}(h_t) \in [-k_t(h_{t-1}), 0], x_{3t}(h_t) \in [-1, 0]\}.$$

Definition 0.22 (Production Function). $Y = \prod_{t=1}^{\infty} Y_t$, *where* $Y_t = \{y \in \mathbb{R}^3 : y_{1t} \leq F(-y_{2t}, -y_{3t})\}$.

TODO: Γ for transition, π for history.

Definition 0.23 (AD model with production shocks).

- *The consumption set is*

$$X(k_0) = \{l \in L : \text{there exists } k_{t+1}(h_t) \text{ and } c_t(h_t) \text{ such that for all } h_t \in H,$$
$$c_t(h_t) + k_{t+1}(h_t) = l_{1t}(h_t), x_{2t} \in [-k_t(h_t), 0], x_{3t} \in [0, 1], c_t(h_t) \geq 0, k_{t+1}(h_t) \geq 0\}.$$

- *The production set is* $Y = \{l \in L : \text{for all } h_t \in H, l_{1t}(h_t) \leq F(-l_{2t}(h_t), -l_{3t}(h_t), z_t(h_t))\}$.

Definition 0.24 (AD with shocks social planner's problem).

$$\arg\max_{k_{t+1}(h_t) \in [0, F(k_t(h_t), 1, z_t(h_t))]} \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \Pi(h_t)\beta^t u(c_t(h_t)),$$

where $c_t(h_t) = F(k_t(h_{t-1}), 1, z_t(h_t)) - k_{t+1}(h_t)$.

Definition 0.25 (ADE with production shocks). *Suppose* $X, Y \subseteq L$. *Then* $(x^*, y^*, p^*) \in X \times Y \times L^*$ *is an ADE if*

- *Representative agent maximizes:*

$$x^* \in \arg\max_{\{x \in X : p^*(x) = 0\}} \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \Pi(h_t)\beta^t u(c_t(h_t)).$$

- *Firms maximize:*

$$y^* \in \arg\max_{y \in Y} p^*(y).$$

  FIXME: need to formalize change in z_t.

- *Market clearing:*

$$x^* = y^*.$$

Definition 0.26 (SMEE with production shocks and bonds). *$(b^*, c^*, k^*, q^*, w^*, R^*)$ is an SMEE given k_0 if*

1.
$$b_t^*, c_t^*, k_{t+1}^* \in \underset{b_t, c_t, k_{t+1}}{\arg\max} \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \Pi(h_t) \beta^t u(c_t(h_t))$$

   such that
$$c_t(h_t) + k_{t+1}(h_t) + \sum_{z \in Z} q_t^*(h_t, z) b_t^*(h_t, z) = R_t^* k_t + w_t^* + b_t^*(h_{t-1}, z_t(h_t)).$$

2. $(k_t^*(h_t), 1) \in \arg\max_{k_t(h_t), n_t(h_t)} z_t F(k_t(h_t), n_t(h_t)) - R_t^*(h_t) k_t(h_t) - w_t^*(h_t) n_t(h_t)$.

3. $b_t^*(h_t) = 0$.

4. $c_t(h_t) + k_{t+1}(h_t) = z_t F(k_t(h_t), 1)$. *[Can be dropped with Walras' law.]*

Note that in SMEE with shocks and bonds, $b_t^*(h_t) = 0$ for all t and all $h_t \in H_t$.
TODO: x^*, y^* is the same as ADE.

Definition 0.27 (SMEE with shocks and no bonds). *$(c^*, k^*, q^*, w^*, R^*)$ is an SMEE given k_0 if*

1.
$$c_t^*(h_t), k_{t+1}^*(h_t, z) \in \underset{c_t(h_t), k_{t+1}(h_t, z)}{\arg\max} \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \Pi(h_t) \beta^t u(c_t(h_t))$$

   such that
$$c_t(h_t) + \sum_{z \in Z} q_t^*(h_t, z) k_{t+1}(h_t, z) = R_t^*(h_t) k_t(h_{t-1}, z_t) + w_t^*(h_t).$$

2. $(k_t^*(h_t), 1) \in \arg\max_{k_t(h_t), n_t(h_t)} f(k_t(h_t), n_t(h_t)) - R_t^*(h_t) k_t(h_t) - w_t^*(h_t) n_t(h_t)$.

3. $c_t(h_t) + k_{t+1}(h_t) = F(k_t(h_t), 1)$. *[Can be dropped with Walras' law.]*

Theorem 0.28. *If (x^*, y^*, p^*) is an ADE, then there exists (q^*, w^*, R^*) such that $(c^*, k^*, q^*, w^*, R^*)$ is an SMEE with shocks and bonds.*

Proof. ADE is equivalent to the SPP. The social planner's problem is,

$$\underset{k_{t+1}(h_t) \in [0, F(k_t(h_t), 1, z_t(h_t))]}{\arg\max} \sum_{t=0}^{\infty} \sum_{h_t \in H_t} \Pi(h_t) \beta^t u(c_t(h_t)),$$

where $c_t(h_t) = F(k_t(h_{t-1}), 1, z_t(h_t)) - k_{t+1}(h_t)$.

The FOC wrt $k_{t+1}(h_t)$ is

$$\Pi(h_t) \beta^t Du(c_t(h_t)) = \sum_{z \in Z} \Pi(h_t, z) \beta^{t+1} Du(c_{t+1}(h_t, z)) D_1 F(k_{t+1}(h_t), 1, z)$$
$$Du(c_t(h_t)) = \beta \sum_{z \in Z} \Gamma_{z_t(h_t), z} Du(c_{t+1}(h_t, z)) D_1 F(k_{t+1}(h_t), 1, z)$$

Similarly, the individual's objective in SMEE is:

$$\underset{k_{t+1}(h_t, z)}{\arg\max} \sum_{t=0}^{\infty} \beta^t \sum_{h_t \in H_t} \Pi(h_t) u(c_t(h_t))$$

where

$$c_t(h_t) = R^*(h_t)k_t(h_t) + w^*(h_t) - \sum_{z \in Z} q_t^*(h_t, z)k_{t+1}(h_t, z).$$

The FOC is wrt $k_{t+1}(h_t, z)$ is

$$\beta^t \Pi(h_t) Du(c_t(h_t)) q_t^*(h_t, z) = \beta^{t+1} \Pi(h_t, z) Du(c_{t+1}(h_t, z)) R^*(h_t)$$
$$Du(c_t(h_t)) q_t^*(h_t, z) = \beta \Gamma_{z_t, z} Du(c_{t+1}(h_t, z)) R^*(h_t)$$

A no arbitrage condition is that $\sum_{z \in Z} q_t^*(h_t, z) = 1$. It follows that

$$Du(c_t(h_t)) = \beta \sum_{z \in Z} \Gamma_{z_t, z} Du(c_{t+1}(h_t, z)) R^*(h_t).$$

Substituting $R^*(h_t) = F_1(k_{t+1}(h_t, z), 1, z)$ gives the same condition as the SPP's FOC. So the problems are equivalent. □

Definition 0.29 (Recursive Competitive Equilibrium with Shocks and no bonds). (g^*, G^*, q^*)

- *Consumer's problem: Let*

$$V(K, a, z; G) = \max_{c, a'_{z'}} [u(c, n) + \beta \sum_{z' \in Z} \Gamma_{zz'} V(G(K, z'), a'_{z'}, z'; G)]$$

  such that

$$c + \sum_{z' \in Z} q^*(z') a'_{z'} = R(K, z)a + w(K, z)$$

  where $R(K, z) = zF_N(K, N)$, $w(K, z) = zF_K(K, N)$ *and* $N = H(K)$.

  Then $(g^*, h^*) = \arg\max_{a', n} \ldots$.

- *Representative agents:* $g^*(z, K, K; G^*) = G^*(z, K)$ *and for all* K.