Log-Linear Approximations

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In economics, an equilibrium of an economy can usually be defined as a vector that satisfies a set of equations. These equations are often hard to solve. This document describes how log-linear approximations of the equations simplify the study of equilibria.

From standard calculus, we know that a wide variety of functions can be locally approximated with linear functions (or rather, affine functions – perhaps this document should be called *log-affine approximations.*). From linear algebra, we also know that linear equations are easy to study, and easy to solve using Gauss-Jordan elimination.

However, in some economics equations such as in growth models, the relevant functions are locally more like exponential functions than linear functions. To get the best of both worlds – closeness of exponential approximations, and tractability of linear approximations – log-linearization transforms the domain with a log function, and then approximates with a linear function.

Definition 1 (Log-linearization)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and let x^* be some point in $\mathbb{R}$. Then the log-linearization $\hat{f}$ of f around x^* is*

$$\hat{f}(\hat{x}) = f(x^*) + x^* f'(x^*)\hat{x}.$$

As the following claim atests, the $\hat{x}$ in the above definition should be interpreted as $\hat{x} = \log x - \log x^*$.

Claim 1. *$f(x) \approx \hat{f}(\hat{x})$, where $\hat{x} = \log x - \log x^*$.*

More precisely, the distance between $f(x)$ and $f(x^*) + x^* f'(x^*)(\log x - \log x^*)$ is less than

$$\max_{c \in [x, x^*]} \frac{1}{2} |f''(c)(x - x^*)^2|.$$

Proof. Let $g(x) = f(e^x)$. Then $g'(x) = e^x f'(e^x)$. Taylor expansion around $\log x^*$ is

$$\begin{aligned} g(x) &\approx g(\log x^*) + g'(\log x^*)(x - \log x^*) \\ &= f(x^*) + e^{\log x^*} f'(x^*)(x - \log x^*) \\ &= f(x^*) + x^* f'(x^*)(x - \log x^*). \end{aligned}$$

Hence $f(x) = g(\log x) \approx f(x^*) + x^* f'(x^*)(\log x - \log x^*)$, as required. □

You can use some algebra to log-linearize these elementary expressions:

Claim 2.

1. *$x \approx x^*(\hat{x} + 1)$.*
2. *$xy \approx x^* y^*(\hat{x} + \hat{y} + 1)$.*
3. *$\frac{x}{y} \approx \frac{x^*}{y^*}(\hat{x} - \hat{y} + 1)$.*
4. *$x^n \approx x^{*n}(n\hat{x} + 1)$.*
5. *$e^x \approx e^{x^*}(1 + x^*\hat{x})$.*

Proof. 1. Putting $f(x) = x$ into the claim gives

$$\begin{aligned} x &\approx x^* + x^* 1(\log x - \log x^*) \\ &= x^* + x^* \hat{x} \\ &= x^*(\hat{x} + 1). \end{aligned}$$

2. Similar.
3. Put $z = y^{-1}$, and evaluate $xz = x^* z^* (\hat{x} + \hat{z} + 1)$.
4. Put $f(x) = x^n$. Then

$$\begin{aligned} x^n &\approx x^{*n} + x^* n x^{*n-1}(\log x - \log x^*) \\ &= x^{*n}(n\hat{x} + 1) \end{aligned}$$

5. Put $f(x) = e^x$. Then

$$\begin{aligned} e^x &\approx e^{x^*} + x^* e^{x^*}(\log x - \log x^*) \\ &= e^{x^*}(1 + x^* \hat{x}). \end{aligned}$$

□

These algebraic compositions can also be constructed:

Claim 3. *Suppose $f, g : \mathbb{R} \to \mathbb{R}$ have log-linearizations $\hat{f}, \hat{g} : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \hat{f}(\hat{x})$. Then*

1. $f(x) + g(x) \approx \hat{f}(\hat{x}) + \hat{g}(\hat{x})$.
2. $f(x)g(x) \approx g(x^*)\hat{f}(\hat{x}) + f(x^*)\hat{g}(\hat{x}) - f(x^*)g(x^*)$.
3. $f(x)^n \approx f(x^*)^{n-1}\hat{f}(n\hat{x})$.
4. $e^{f(x)} \approx e^{f(x^*)}[\hat{f}(\hat{x}) + 1 - f(x^*)]$.

Proof. 1. Set $h(x) = f(x) + g(x)$. Then

$$\begin{aligned} h(x) &\approx f(x^*) + g(x^*) + x^*(f'(x^*) + g'(x^*))(\log x - \log x^*) \\ &= [f(x^*) + x^* f'(x^*)\hat{x}] + [g(x^*) + x^* g'(x^*)\hat{x}] \\ &= \hat{f}(\hat{x}) + \hat{g}(\hat{x}). \end{aligned}$$

2. Set $h(x) = f(x)g(x)$. Then $h'(x^*) = f(x^*)g'(x^*) + f'(x^*)g(x^*)$, so

$$\begin{aligned} h(x) &\approx f(x^*)g(x^*) + x^*[f(x^*)g'(x^*) + f'(x^*)g(x^*)]\hat{x} \\ &= f(x^*)[g(x^*) + x^* g'(x^*)\hat{x}] + g(x^*)[f(x^*) + x^* f'(x^*)\hat{x}] - f(x^*)g(x^*) \\ &= g(x^*)\hat{f}(\hat{x}) + f(x^*)\hat{g}(\hat{x}) - f(x^*)g(x^*). \end{aligned}$$

3. Set $h(x) = f(x)^n$. Then

$$\begin{aligned} h(x) &\approx f(x^*)^n + x^* n f(x^*)^{n-1} f'(x^*)\hat{x} \\ &= f(x^*)^{n-1}[f(x^*) + x^* f'(x^*) n\hat{x}] \\ &= f(x^*)^{n-1}\hat{f}(n\hat{x}). \end{aligned}$$

4. Set $h(x) = e^{f(x)}$.

$$\begin{aligned} h(x) &\approx e^{f(x^*)} + x^* e^{f(x^*)} f'(x^*)\hat{x} \\ &= e^{f(x^*)}[1 + x^* f'(x^*)\hat{x}] \\ &= e^{f(x^*)}[\hat{f}(\hat{x}) + 1 - f(x^*)]. \end{aligned}$$

□

The first claim can be generalized to provide log-linearizations of multi-variate functions.

Claim 4.

$$f(x, y) \approx f(x^*, y^*) + x^* D_1 f(x^*, y^*)\hat{x} + y^* D_2 f(x^*, y^*)\hat{y},$$

where $\hat{x} = \log x - \log x^*$ *and* $\hat{y} = \log y - \log y^*$.

Proof. Let $g(x, y) = f(e^x, e^y)$. Then $Dg(x, y) = e^x D_1 f(e^x, e^y) + e^y D_2 f(e^x, e^y)$. The Taylor expansion around $(\log x^*, \log y^*)$ is

$$\begin{aligned} g(x, y) &\approx g(\log x^*, \log y^*) + Dg(\log x^*, \log y^*)(x - \log x^*, y - \log y^*)^T \\ &= f(x^*, y^*) + e^{\log x^*} D_1 f(x^*, y^*)(x - \log x^*) + e^{\log y^*} D_2 f(x^*, y^*)(y - \log y^*) \\ &= f(x^*, y^*) + x^* D_1 f(x^*, y^*)(x - \log x^*) + y^* D_2 f(x^*, y^*)(y - \log y^*). \end{aligned}$$

Hence $f(x, y) = g(\log x, \log y) \approx f(x^*, y^*) + x^* D_1 f(x^*, y^*)(\log x - \log x^*) + y^* D_2 f(x^*, y^*)(\log y - \log y^*)$, as required. □

These multi-variate functions have the following log-linearizations:

Claim 5. *Suppose* $f, g : \mathbb{R} \to \mathbb{R}$ *have log-linearizations of* $\hat{f}, \hat{g}$ *such that* $f(x) = \hat{f}(\hat{x})$.

1. $h(x, y) = f(x) + g(y) \approx \hat{f}(\hat{x}) + \hat{g}(\hat{x})$.

2.

$$\begin{aligned} h(x, y) = f(x)g(y) &= f(x^*)g(y^*) + x^* f'(x^*)g(y^*)\hat{x} + y^* f(x^*)g'(x^*)\hat{y} \\ &= f(x^*)\hat{g}(\hat{y}) + g(y^*)\hat{f}(\hat{x}) - f(x^*)g(y^*). \end{aligned}$$

3. $h(x, y) = x^m y^n \approx x^{*m} y^{*n}(1 + m\hat{x} + n\hat{y})$

4. $h(x, y) = e^{x+y} \approx e^{x^* + y^*}(1 + x^*\hat{x} + y^*\hat{y})$.

Proof. 1. Trivial.

2. Set $h(x, y) = f(x)g(y)$. Then

$$\begin{aligned} h(x, y) &\approx h(x^*, y^*) + x^* D_1 h(x^*, y^*)\hat{x} + y^* D_2 h(x^*, y^*)\hat{y} \\ &= f(x^*)g(y^*) + x^* f'(x^*)g(y^*)\hat{x} + y^* f(x^*)g'(x^*)\hat{y} \\ &= f(x^*)\hat{g}(\hat{y}) + g(y^*)\hat{f}(\hat{x}) - f(x^*)g(y^*). \end{aligned}$$

3. $h(x, y) = x^m y^n = x^{*m} y^{*n} + x^* m x^{*m-1} y^{*n}\hat{x} + y^* x^{*m} n y^{*n-1}\hat{y} = x^{*m} y^{*n}(1 + m\hat{x} + n\hat{y})$.

4. $h(x, y) = e^{x+y} = e^x e^y = e^{x^*} e^{y^*} + x^* e^{x^*} e^{y^*}\hat{x} + y^* e^{x^*} e^{y^*}\hat{y} = e^{x^* + y^*}(1 + x^*\hat{x} + y^*\hat{y})$.

□

Examples:

- $Y_t = C_t + G_t$ can be log-linearized to $\hat{Y}_t = \frac{C^*}{Y^*}\hat{C}_t + \frac{G^*}{Y^*}\hat{G}_t$:

$$
\begin{aligned}
Y_t &= C_t + G_t \\
Y^*(\hat{Y}_t + 1) &= C^*(\hat{C}_t + 1) + G^*(\hat{G}_t + 1) \\
Y^*\hat{Y}_t + Y^* &= C^*\hat{C}_t + G^*\hat{G}_t + (C^* + G^*) \\
Y^*\hat{Y}_t &= C^*\hat{C}_t + G^*\hat{G}_t \\
\hat{Y}_t &= \frac{C^*}{Y^*}\hat{C}_t + \frac{G^*}{Y^*}\hat{G}_t
\end{aligned}
$$

Note that the cancelation occured because $Y^* = C^* + G^*$.