Summary of General Equilibrium

Andrew Clausen <clausen@econ.upenn.edu> *

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Contents

TODO: generalize to Banach spaces (in appendix?), consumption with boundary, investigate transversality, E-B box, clean up fixed point section, prove boundedness of production function, prove Farkas, SHT and KT theorems.

1 Notation

This text has a few idiosyncracities:

- $\tilde{x}_h^g = \sum_{h \in H} x^g$.
- If $x \in \mathbb{R}^G$ then $x^{\setminus G}$ means $(x^1, \ldots, x^{g-1}, x^{g+1}, x^G)$ and $(x^{\setminus G}, a)$ means $(x^1, \ldots, x^{g-1}, a, x^{g+1}, \ldots, x^G)$.

*Nirav Mehta contributed two of the diagrams, and he and Seth Richards found numerous mistakes, especially in the proof of the properness of the smooth equilibrium manifold.

- If r is a matrix, then $r(s)$ denotes the s^{th} row or column of the matrix. Which should be clear from context. Similarly, $r(\backslash s)$ denotes the entire matrix with the s^{th} row or column removed.
- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *increasing* if
  - $x > y$ implies $f(x) \geq f(y)$, and
  - $x \gg y$ implies $f(x) > f(y)$.
- A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *strictly increasing* if
  - $x > y$ implies $f(x) > f(y)$, and
  - $x \gg y$ implies $f(x) \gg f(y)$.

2 Optimization

Definition 2.1 (Convex cone). *If $a^1, \ldots, a^m \in \mathbb{R}^n$, then*

$$\text{cone}(a^1, \ldots, a^m) = \{A\lambda : \lambda \in \mathbb{R}^m_+\} = \{\lambda^1 a^1 + \cdots + \lambda^m a^m : \lambda \in \mathbb{R}^m_+\}.$$

Definition 2.2 (Hyperplane). *If $q^T \in \mathbb{R}^n \backslash \{0\}$, then the hyperplane $H(q)$ orthogonal to q is*

$$H(q) = \{x \in \mathbb{R}^n : qx = 0\}.$$

Definition 2.3 (Halfspace). *If $H(q)$ is a hyperplane, then the positive halfspace $H^+(q)$ above the hyperplane $H(q)$ is $H^+(q) = \{x \in \mathbb{R}^n : qx \geq 0\}$. Similarly, the negative halfspace $H^-(q) = -H^+(q)$ lies below the hyperplane.*

Definition 2.4 (Supporting hyperplane). *A hyperplane $H(\lambda) + p$ supports a set $X \subset \mathbb{R}^n$ if some $X \subseteq H^-(\lambda) + p$ and $(H(\lambda) + p) \cap \text{cl}(X) \neq \emptyset$.*

Theorem 2.5 (Supporting hyperplane theorem). *If $X \subset \mathbb{R}^n$ is convex and $x \in \text{bd}(X)$, then there is a supporting hyperplane $H(\lambda) + x$ of X.*

Lemma 2.6 (Farkas lemma). *Suppose that $a^1, \ldots, a^m \in \mathbb{R}^n \backslash \{0\}$. If $z \in \mathbb{R}^n$, then*

- *either $z^* \in \text{cone}(a^1, \ldots, a^m)$,*
- *or $z^* \notin \text{cone}(a^1, \ldots, a^m)$ and there exists some $q^{*T} \in \mathbb{R}^n \backslash \{0\}$ such that*
  - *$z^* \in \text{int}(H^+(q^*))$*
  - *and $\text{cone}(a^1, \ldots, a^m) \subseteq H^-(q^*)$.*

Theorem 2.7 (Lagrange multipliers). *Let X be some open subset of $\mathbb{R}^n$. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$ be differentiable. Define $L : X \times \mathbb{R}^m \to \mathbb{R}$ as $L(x, \lambda) = f(x) + \lambda^T g(x)$. If*

$$x^* \in \underset{x \in g^{-1}(0)}{\arg\max}\ f(x),$$

and $Dg(x^*)$ has full rank, then there exists $\lambda^* \in \mathbb{R}^m$ such that $D_x L(x^*, \lambda^*) = 0$.

Definition 2.8 (Constraint qualification). *We say that $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$ satisfies constraint qualification at $x \in X$ if either*

- $\det(Dg(x^*)) \neq 0$.

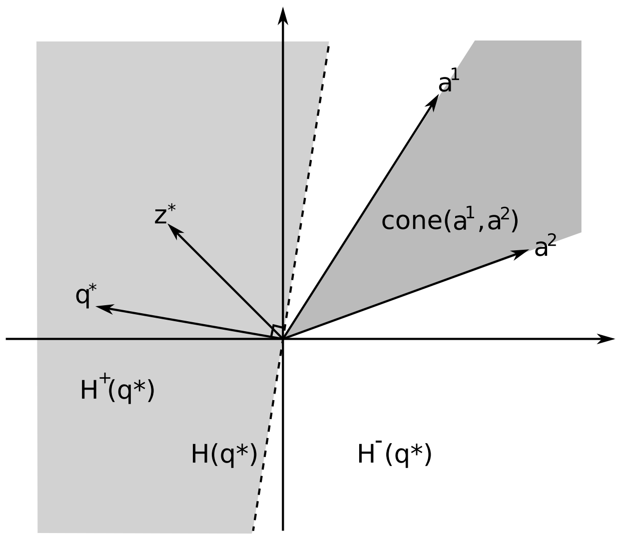

Figure 1: Farkas' lemma says that if z^* is outside of the cone, then there is a hyperplane running through the origin with z^* strictly on one side, and the cone weakly on the other side.

- *g is linear.*
- *(Slater's condition) f and g are concave and there is some $x^* \in X$ such that $g(x^*) \gg 0$.*

The Karush-Kuhn-Tucker[1] conditions give first order characterizations of solutions to inequality constrained optimization problems.

Theorem 2.9 (Karush-Kuhn-Tucker neccessary conditions). *Let X be some open subset of $\mathbb{R}^n$. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$ be differentiable. Let $L : X \times \mathbb{R}^m \to \mathbb{R}$ be $L(x, \lambda) = f(x) + \lambda^T g(x)$. If*

$$x^* \in \arg\max_{x \in g^{-1}(\mathbb{R}^m_+)} f(x),$$

and g satisfies constraint qualification at x^*, then there exists $\lambda^* \in \mathbb{R}^m_+$ such that

- *First order condition: $D_x L(x^*, \lambda^*) = 0$.*
- *Complementary slackness: $\lambda_i^* g_i(x^*) = 0$ for all $i \in \{1, \ldots, m\}$.*

Theorem 2.10 (Karush-Kuhn-Tucker sufficient conditions). *Let X be some open subset of $\mathbb{R}^n$. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$ be differentiable and quasiconcave. Let $L : X \times \mathbb{R}^m \to \mathbb{R}$ be $L(x, \lambda) = f(x) + \lambda^T g(x)$. If there exists $x^* \in X$ and $\lambda^* \in \mathbb{R}^m_+$ such that*

- *First order condition: $D_x L(x^*, \lambda^*) = 0$.*
- *Complementary slackness: $\lambda_i^* g_i(x^*) = 0$ for all $i \in \{1, \ldots, m\}$.*

and either $Df(x^*) \neq 0$ or f is concave, then

$$x^* \in \arg\max_{x \in g^{-1}(\mathbb{R}^m_+)} f(x).$$

[1] These are often called the Kuhn-Tucker conditions. However, Albert Tucker says in Transcript 39 (Career Part 2) of *The Princeton Mathematics Community in the 1930s* that "we now know that it should be called Karush-Kuhn-Tucker theory because Bill Karush had anticipated what we did in 1950 in his master's thesis at Chicago about 1940."

3 Fixed Points

There are two obvious ways to define the inverse of a correspondence.

Definition 3.1 (Upper inverse). *If* $\Gamma : X \rightrightarrows Y$ *is a correspondence, then* $\Gamma^{-1}(A) = \{x \in X : \Gamma(x) \subseteq A\}$.

Definition 3.2 (Lower inverse). *If* $\Gamma : X \rightrightarrows Y$ *is a correspondence, then* $\Gamma_{-1}(A) = \{x \in X : \Gamma(x) \cap A \neq \emptyset\}$.

If Γ is non-empty valued, then $\Gamma^{-1}(A) \subseteq \Gamma_{-1}(A)$, which is a little confusing (since *upper* implies *smaller*).

Definition 3.3 (Upper hemi-continuous correspondence). *Equivalent definitions for* $\Gamma : X \rightrightarrows Y$ *is UHC:*

1. *For all open sets* $U \subseteq Y$, $\Gamma^{-1}(U)$ *is open.*
2. *(UHC at* $x \in X$*) For all open sets* $U \subseteq Y$ *with* $x \in \Gamma^{-1}(U)$, $\Gamma^{-1}(U)$ *is open.*

Definition 3.4 (Lower hemi-continuous correspondence). *Equivalent definitions for* $\Gamma : X \rightrightarrows Y$ *is LHC:*

1. *For all open sets* $U \subseteq Y$, $\Gamma_{-1}(U)$ *is open.*
2. *(LHC at* $x \in X$*) For all open sets* $U \subseteq Y$ *with* $x \in \Gamma_{-1}(U)$, $\Gamma_{-1}(U)$ *is open.*
3. *(LHC at* $x \in X$*) For all sequences* $x_k \to x$ *and all* $y \in \Gamma(x)$, *there exists* $y_k \to y$ *with* $y_k \in \Gamma(x_k)$.

Definition 3.5 (Graph). *If* $\Gamma : X \rightrightarrows Y$, *then* $\text{graph}(\Gamma) = \{(x, y) \in X \times Y : y \in \Gamma(x)\}$.

Definition 3.6 (Upper sequence property). *(non-standard) If* $\Gamma : X \rightrightarrows Y$ *has the upper sequence property if all sequences* $(x_n, y_n) \in \text{graph}(\Gamma)$ *for which* $x_n \to x$ *must have a convergent subsequence with limit* $(x, y) \in \text{graph}(\Gamma)$.

Proposition 3.7. *If* Γ *has the upper sequence property, it is UHC. If* Γ *is UHC and compact-valued, then it has the upper sequence property.*

Proposition 3.8. *If* $\Gamma : X \rightrightarrows Y$ *is UHC and compact-valued, and* $K \subseteq X$ *is compact, then* $\Gamma(K)$ *is compact.*

This proposition explains why correspondences with closed graphs are sometimes called *sequentially UHC*.

Proposition 3.9. *If* $\text{graph}(\Gamma)$ *is closed and* Y *is compact, then* $\Gamma : X \rightrightarrows Y$ *is UHC.*

Proposition 3.10. *If* $\Gamma : X \rightrightarrows Y$ *is UHC and closed valued, then* $\text{graph}(\Gamma)$ *is closed.*

Definition 3.11 (Hausdorff metric). *Let* (X, d) *be a metric space. Then* $(2^X, d_H)$ *is the metric space with*

$$d_H(A, B) = \max\{\omega(A, B), \omega(B, A)\}$$
$$\omega(A, B) = \sup_{a \in A} d(a, B).$$

Theorem 3.12. *Let* $\Gamma : X \rightrightarrows Y$ *be a correspondence. For all* $x \in X$,

1. Γ *is UHC at* x *if and only if for all* $x' \in X$ *with* $d_H(\Gamma(x'), \Gamma(x)) < \epsilon$, $\omega(\Gamma(x'), \Gamma(x)) < \epsilon$.
2. Γ *is LHC at* x *if and only if for all* $x' \in X$ *with* $d_H(\Gamma(x'), \Gamma(x)) < \epsilon$, $\omega(\Gamma(x), \Gamma(x')) < \epsilon$.

Corollary 3.13 (Hausdorff characterization of continuous correspondences). $\Gamma : X \rightrightarrows Y$ *is continuous in* $X \to 2^Y$ *iff* Γ *is both UHC and LHC.*

Theorem 3.14 (Maximum theorem (Berge)). *If* $\Gamma : \Theta \rightrightarrows X$ *is non-empty and compact-valued and* $f : \Theta \times X \to \mathbb{R}$ *is continuous, then*

1. *$\sigma(\theta) = \arg\max_{x \in \Gamma(\theta)} f(x;\theta)$ is non-empty and compact valued.*

2. *If Γ is continuous and X is compact then σ is UHC and $F(\theta) = f(\sigma(\theta), \theta)$ is continuous.*

Definition 3.15 (Fixed point). *Let $\Gamma : X \rightrightarrows X$ be a correspondence. A point $x \in X$ is a fixed point of Γ if $x \in \Gamma(x)$.*

Theorem 3.16 (Kakutani fixed point theorem). *If $\Gamma : X \rightrightarrows X$ is UHC, non-empty and convex-valued, and $X \subseteq \mathbb{R}^n$ is non-empty, compact and convex, then Γ has a fixed point.*

4 Parametric Smooth Analysis

TODO: rant about implicit function theorem.

The parametric smooth analysis framework consists of:

- **a space** $H = \Xi \times \Theta \subseteq \mathbb{R}^J \times \mathbb{R}^K$ where Ξ and Θ are open (relative to $\mathbb{R}^J$ and $\mathbb{R}^K$). A typical element of H is denoted either η or (ξ, θ). Θ is the *parameter space* and Ξ is the *value space*.

- **a smooth function** $\Phi : H \to \mathbb{R}^L$, where Φ is C^1. (We are especially interested in the cases where $J = L$.)

Note that the particular metric space (or more generally, topological space) we are working with is important. A set may be closed inside H but not inside $\mathbb{R}^J \times \mathbb{R}^K$.

Definition 4.1 (Solution manifold). *$M = \Phi^{-1}(0) = \{\eta \in H : \Phi(\eta) = 0\}$.*

Definition 4.2 (Solution projection). *$\pi : M \to \Theta$ is the function with $\pi(\xi, \theta) = \theta$.*

Definition 4.3 (Open mapping). *A function $f : X \to Y$ is an open mapping if $f(U) \subseteq Y$ is open for all open sets $U \subseteq X$.*

Definition 4.4 (Closed mapping). *A function $f : X \to Y$ is a closed mapping if $f(U) \subseteq Y$ is closed for all closed sets $U \subseteq X$.*

Note that an open mapping need not be closed, and vice versa. (The two are equivalent for bijections.)

Theorem 4.5 (Projections are open mappings). *The function $\pi : X \times Y \to X$ defined by $\pi(x, y) = x$ is an open mapping.*

Proof. This is a special case of the open mapping theorem, which asserts all surjective continuous linear functionals between Banach spaces (i.e. complete spaces) are open mappings. □

Definition 4.6 (Proper function). *A function $f : X \to Y$ is proper if f is continuous, and for all compact $\bar{Y} \subseteq Y$, $f^{-1}(\bar{Y}) \subseteq X$ is compact.*

Figure 2 gives an example of a set M for which π is proper. Figure 3 gives two examples in which π is not proper.

Theorem 4.7 (Properness theorem). *If $f : X \to Y$ is proper, then f is a closed mapping.*

Proof. Suppose X' is closed and $y^* \in \operatorname{bd} f(X')$. Then there exists some sequence $x_k \in X'$ with $f(x_k) \to y^*$. Notice that $Y^* = \{y^*, f(x_k)\}$ is compact since every sequence in Y^* has a convergent subsequence. (All sequences in Y^* either have constant subsequences, or subsequences that converge to y^*.) So $X^* = f^{-1}(Y^*)$ is compact, implying x_k has a convergent subsequence $\hat{x}_k$ in X^*. By continuity of f, it follows that $f(\hat{x}_k) \to y^*$. Since X' is closed, $\lim \hat{x}_k \in X'$, and the conclusion that $y^* \in f(X')$ follows. □

Definition 4.8 (Generic set). *A set U is generic if it is open, and its complement U^c has zero measure.*

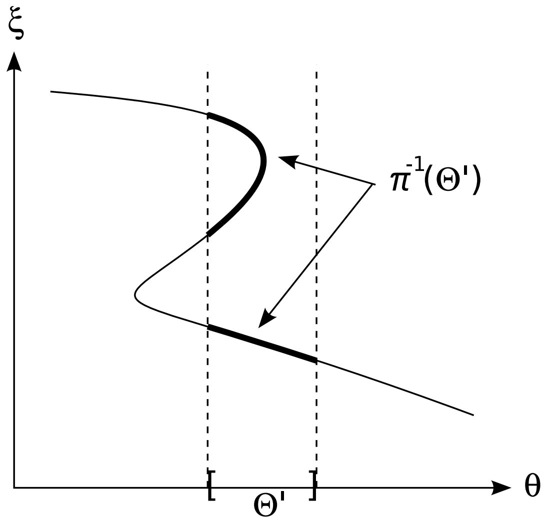

Figure 2: For $\pi : \Xi \times \Theta \to \Theta$ to be proper, the preimage $\pi^{-1}(\Theta')$ of every compact set $\Theta' \subseteq \Theta$ must be compact. In the above example, $\pi^{-1}(\Theta')$ is compact as required. Note that the domain of π is the set of points on the curve, M (which is a subspace of the entire plane), and the codomain is a subset of the horizontal axis. But, this does not matter very much, since compactness is a property that does not change when different ambient spaces are considered.

Definition 4.9 (Critical points). *Let M be the solution manifold of some function $\Phi : H \to \mathbb{R}^L$. The set of critical points of Φ is $M^c = \{(\xi, \theta) \in M : \text{rank}(D_1\Phi(\xi, \theta)) < L\}$.*

Definition 4.10 (Critical values). *The set of critical values of a solution manifold M is $\Theta^c = \pi(M^c)$.*

Definition 4.11 (Regular points). *The set of regular values of a solution manifold M is $M^r = M \backslash M^c$.*

Definition 4.12 (Regular values). *The set of regular values of a solution manifold is $\Theta^r = \Theta \backslash \Theta^c$.*

Theorem 4.13 (Closedness theorem). *If the manifold projection $\pi : M \to \Theta$ is proper, then Θ^c is closed ($\iff$ Θ^r is open).*

Proof. If $J < L$, then $\text{rank}(D_\xi\Phi(\xi, \theta)) < L$ trivially, so $\Theta^c = \pi(M)$. $M = \Phi^{-1}(0)$ is clearly closed, and the Properness theorem gives that $\pi(M)$ is closed.

If $J = L$, then $D_\xi\Phi(\xi, \theta)$ is a square matrix, and has rank smaller than L if and only if $\det(D_\xi\Phi(\xi, \theta)) = 0$. Let $f(\xi, \theta) = \det(D_\xi\Phi(\xi, \theta))$. Now $\Theta^c = \pi(M \cap f^{-1}(0))$. f is continuous means $f^{-1}(0)$, and hence $M \cap f^{-1}(0)$ is closed. The Properness theorem gives that $\pi(M) \cap f^{-1}(0)$ is closed. □

The transversality theorem depicted in Figure 4 asserts that if the total derivative of Φ has full rank, then the set of critical points Θ^c (where the *partial* derivative with respect to ξ does not have full rank) has zero measure.

Theorem 4.14 (Transversality theorem). *If $\Phi : \Xi \times \Theta \to \mathbb{R}^L$ has $\text{rank}(D\Phi) = L$ everywhere, then Θ^c has zero measure ($\iff$ Θ^r has full measure).*

Proof. Dave recommends Milnor, section 2, for those who are interested. □

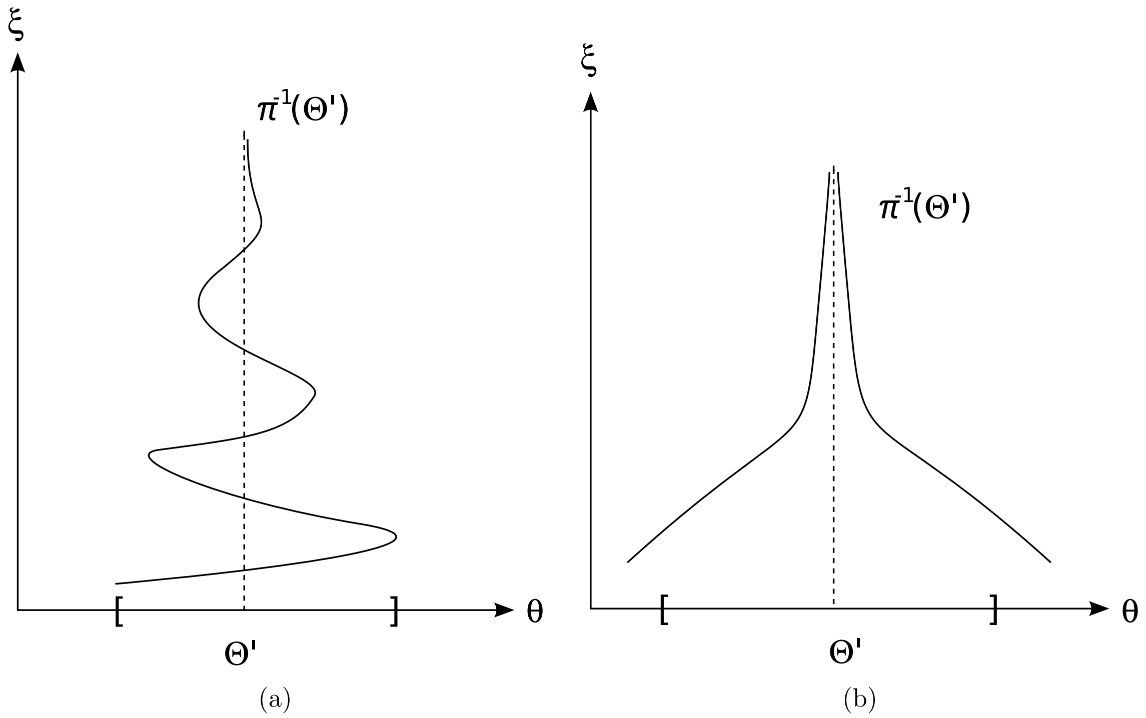

Figure 3: π is not proper in either of these examples, since each indicated Θ' is compact and has unbounded preimage $\pi^{-1}(\Theta')$.

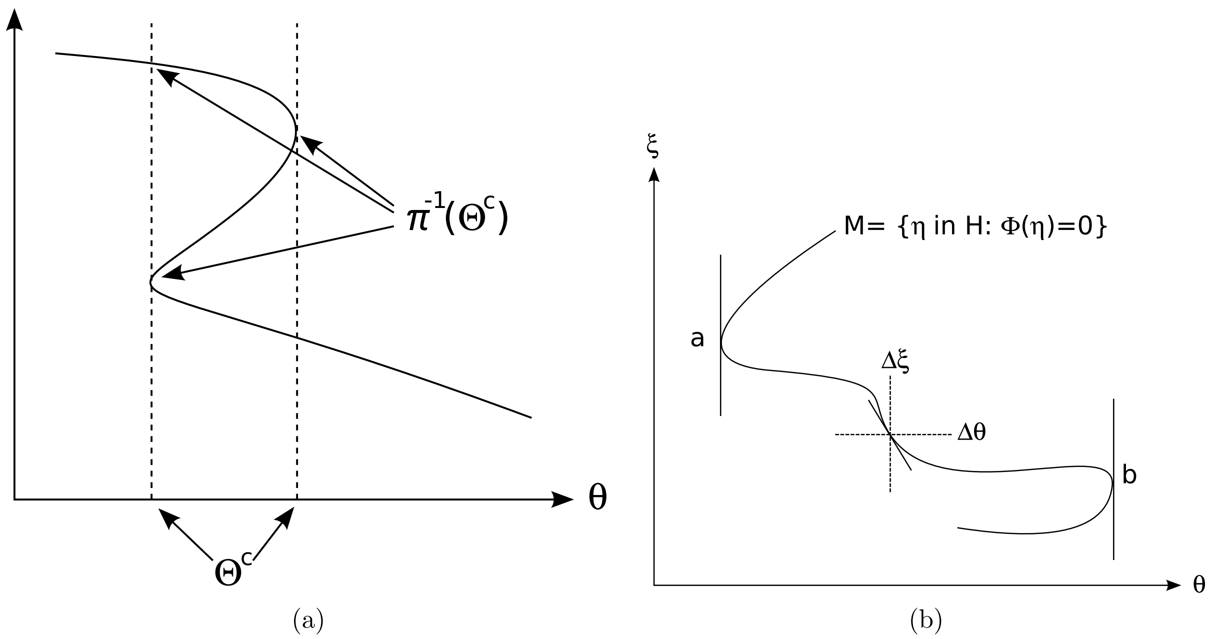

Figure 4: π^{-1} maps the critical values Θ^c to the set $\pi^{-1}(\Theta^c)$ which contains all critical points — and possibly some regular points as well, depicted in (a). The transverality theorem says that the set of critical values is small (has measure zero), and hence almost all points have a derivative, as depicted in (b).

If $M = \Phi^{-1}(0)$ meets the conditions of the transversality theorem and the closedness theorem, then Θ^r is generic. This can be used in two useful ways:

- If $J = L$ (i.e. the number of endogeneous variables equals the number of parameters), then we can show M has various properties generically. The remainder of this section gives an example of finite local uniqueness of solutions.
- If $J < L$, then we can show that M has zero measure. That is, if $J < L$, then every point in M is a critical point. But Θ^r is generic, so this implies $\pi^{-1}(\Theta^r) = \emptyset$. That is, all the regular points are regular trivially – because the manifold is empty there. This is useful because we can construct M to be things like "the set of financial equilibria that are Pareto efficient", and then prove that this set is very small.

Figure 5 depicts the implicit function theorem.

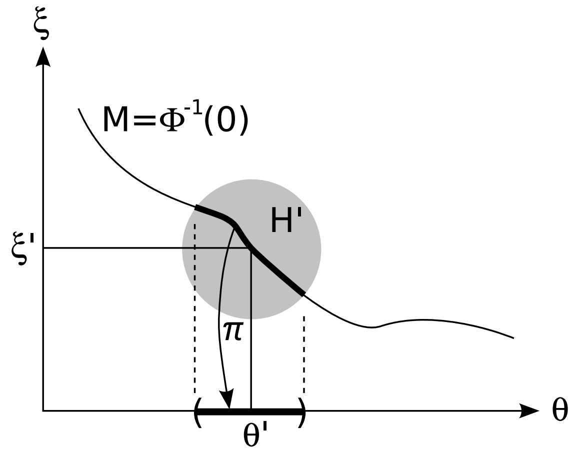

Figure 5: The implicit function theorem.

Theorem 4.15 (Implicit function theorem). *Let M be the solution manifold of a function $\Phi : \Xi \times \Theta \to \mathbb{R}^J$, where $\dim(\Xi) = J$. If $(\xi', \theta') \in M^r$, then there is some open set $H' \subseteq H = \Xi \times \Theta$ containing (ξ', θ') such that the restricted manifold projection $\pi|_{H'}$ is a diffeomorphism onto its range.*

Corollary 4.16. *Let $f : \pi(H') \to M \cap H'$ be $f = (\pi|_{H'})^{-1}$. Under the conditions of the implicit function theorem,*

$$Df_1(\theta) = -[(D_1\Phi)(f(\theta))]^{-1}(D_2\Phi)(f(\theta))$$

for all $\theta \in \pi(H')$.

Proof. Since $(\Phi \cdot f)(\theta) = 0$ for all $x \in \pi(H')$, it follows that

$$\begin{aligned} D(\Phi \circ f)(\theta) &= 0 \\ D\Phi(f(\theta))Df(\theta) &= 0 \\ D_1\Phi(f(\theta))Df_1(\theta) + D_2\Phi(f(\theta))I &= 0 \\ Df_1(\theta) &= -[(D_1\Phi)(f(\theta))]^{-1}(D_2\Phi)(f(\theta)). \end{aligned}$$

□

Definition 4.17 (Finite locally unique correspondence). *A correspondence $f : \Theta \rightrightarrows M$ into $M \subseteq H$ is finite locally unique at θ if*

- *$f(\theta)$ is finite.*
- *There is an open cover $\{U_\eta\}$ in H of $f(\theta)$ such that $f^{-1}|_{U_\eta}$ is a homeomorphism onto its range for all $\eta \in f(\theta)$.*

If each $f^{-1}|_{U_\eta}$ is a diffeomorphism onto its range, then f is differentiably finite locally unique.

Theorem 4.18 (Stack of records theorem). *If the manifold projection $\pi : M \to \Theta$ is proper then π^{-1} is differentiably finite locally unique at all $\theta \in \Theta^r$.*

Proof. This proof first constructs the required open cover of $\pi^{-1}(\theta)$, and then verifies $\pi^{-1}(\theta)$ is finite.

Firstly, by definition of Θ^r, every $\theta \in \Theta^r$ has $\pi^{-1}(\theta) \subseteq M^r$. So the implicit function theorem applies to all of $\pi^{-1}(\theta)$, giving an open cover $\mathcal{U} = \{U_\xi\}$ in H of $\pi^{-1}(\theta)$ such that for each $(\xi, \theta) \in \pi^{-1}$, the projection $\pi|_{U_\xi}$ is a diffeomorphism onto its range.

Secondly, let $\mathcal{U}^*$ be any disjoint open cover of $\pi^{-1}(\theta)$ relative to M. (The previous part clearly implies such a $\mathcal{U}^*$ exists.) The singleton $\{\theta\}$ is compact, so $\pi^{-1}(\theta)$ is compact by the properness assumption. This implies $\mathcal{U}^*$ has a finite subcover. But $\mathcal{U}^*$ is disjoint and has no strict subcover, so $\mathcal{U}$ must be finite. Hence $\pi^{-1}(\theta)$ is finite. ☐

5 Models

Definition 5.1 (Locally non-satiated utility function). *A utility function $u : X \to \mathbb{R}$ is locally non-satiated if for every $x \in X$ and every open neighbourhood N of x, there exists some $x' \in N$ with $u(x') > u(x)$.*

Definition 5.2 (Locally non-satiated spot-by-spot). *A utility function $u : X \to \mathbb{R}$ is locally non-satiated spot-by-spot if for every $x \in X$, every spot $s \in \mathcal{S}$, and every open neighbourhood $N(s)$ of $x(s)$ in $X(s)$, there is some $x'(s) \in N(s)$ such that $u((x(\backslash s), x'(s))) > u(x)$.*

Note that spot-by-spot local non-satiation implies local non-satiation. Provided that the domain is well-behaved (eg: open, or open above), increasing implies locally non-satiated.

The basic model is primarily for existence. The consumption set includes its boundary, so it can be easily compactified for Kakutani's fixed point theorem. The utility functions are continuous for the Berge's maximum theorem, and quasi-concavity implies the demand correspondence are convex-valued. Continuity, local non-satiation and quasiconcavity together imply that locally non-binding constraints can be dropped by the constraint relaxation lemma, which is useful for compactification in the existence proof. Endowments in the interior guarantee that the endowments have positive value pe. This is useful in the existence proof for constructing a sequence "scaled-down" versions of a target consumption vector that converges to the target.

Definition 5.3 (Basic model). *(Primarily for existence)*

- *The consumption set is $X_h = \mathbb{R}^G_+$.*
- *Utility functions $u_h : X_h \to \mathbb{R}$ are C^0, locally non-satiated and quasi-concave.*
- *Endowments have $e_h \gg 0$.*

Definition 5.4 (Standard model). *(For Edgeworth-Bowley box)*

- *The consumption set is $X_h = \mathbb{R}^G_+$.*
- *Utility functions $u_h : X_h \to \mathbb{R}$ are C^1, strictly increasing and strictly quasi-concave.*
- *Endowments have $e_h > 0$.*

The smooth model is designed to fit into the smooth analysis framework where the set of equilibria parameterized by endowments is a smooth manifold. This framework is useful for proving finite local uniqueness of equilibria, and more generally, verifying generic properties. Equilibria are completely characterized by a differential equation. The consumption sets are open in $\mathbb{R}^G$ so that the utility functions can be differentiable – C^2 in fact – on their domains. The differentiably increasing and quasi-concavity requirements give first and second order conditions are used to prove the equilibrium manifold has full rank everywhere. The boundary condition is used in the properness proof to ensure that there is a convergent sequence with a limit inside X_h. The condition on the endowments is redundant, as endowments must lie in the consumption set.

Definition 5.5 (Smooth model). *(For finite local uniqueness)*

- *The consumption set is* $X_h = \mathbb{R}^G_{++}$.
- *Utility functions* $u_h : X_h \to \mathbb{R}$ *are* C^2*, differentiably strictly increasing, differentiably strictly quasi-concave, and satisfy the boundary coundition that* $\mathrm{cl}(U_h(x)) \subseteq X_h$.
- *Endowments have* $e_h \in X_h$*. (i.e.* $e_h \gg 0$.*)*

Definition 5.6 (Smooth model with boundary). *(For finite local uniqueness)*

- *The consumption set is* $X_h = \mathbb{R}^G_{+}$.
- $\underline{X}_h = \{x_h \in \mathbb{R}^G : x_h^g > -\epsilon\}$ *for some* $\epsilon > 0$.
- *Utility functions* $u_h : \underline{X}_h \to \mathbb{R}$ *are* C^2*, differentiably strictly increasing, differentiably strictly quasi-concave, and satisfies the boundary coundition that for all* $x \in \mathrm{int}(X_h)$*,* $\mathrm{cl}(U(x)) \subseteq \underline{X}_h$.
- $e_h \in \mathrm{int}(X_h)$.

6 Equilibrium

Definition 6.1 (Budget correspondence). *The budget correspondence* $B : X^*_+ \times X \rightrightarrows X$ *is*

$$B(p, e) = \{x \in X : p(e - x) = 0\}.$$

Definition 6.2 (Weak budget correspondence). *The weak budget correspondence* $B^{\leq} : X^*_+ \times X \rightrightarrows X$ *is*

$$B^{\leq}(p, e) = \{x \in X : p(e - x) \geq 0\}.$$

Theorem 6.3 (Continuity of budget correspondence). B *and* $B^{\leq}$ *are continuous.*

Definition 6.4 (General equilibrium). *The triple* (x^*, p^*) *of consumptionn and prices is a general equilibrium if*

- *Utility maximization:*
$$x_h^* \in \underset{x_h \in B(p^*, e_h)}{\arg\max}\ u_h(x_h).$$
- *Materials balance:* $\tilde{x}^* = \tilde{e}$.

If $B(p^*, e_h)$ *is replaced with* $B^{\leq}(p^*, e_h)$*, then* (x^*, y^*, p^*) *is a weak general equilibrium.*

Definition 6.5 (General equilibrium with transfers). *The triple* (x^*, p^*, τ^*) *of consumption, prices and transfers is a general equilibrium if*

- *Utility maximization:*
$$x_h^* \in \operatorname*{arg\,max}_{x_h \in B(p^*, \tau^*, e_h)} u_h(x_h),$$
where $B(p^*, \tau^*, e_h) = \{x \in X_h : p^* x = p^* e_h + \tau^*\}$.

- *Transfers balance:* $\tilde{\tau}^* = 0$.

- *Materials balance:* $\tilde{x}^* = \tilde{e}$.

Again, weak general equilibrium with transfers is analogous.

Theorem 6.6 (Price determinacy). *If* $p^{*G} > 0$, *then* (x^*, p^*) *is a GE if and only if* $(x^*, (p^{*\backslash G}/p^{*G}, 1))$ *is a GE.*

Without loss of generality, can drop one of the market clearing constraints.

Theorem 6.7 (Walras law). *If* $\tilde{e}_{\backslash G} = \tilde{x}^*_{\backslash G}$, *and*
$$x_h^* \in \operatorname*{arg\,max}_{x_h \in B(p, e_h)} u_h(x_h)$$
for some p *with* $p_G > 0$, *then* $\tilde{x}^* = \tilde{e}$.

Proof. Since each $x_h^* \in B(p, e_h)$, we have $p(e_h - x_h^*) = 0$. Summing over all households gives $p(\tilde{e} - \tilde{x}^*) = 0$. Now if $\tilde{e}_{\backslash G} = \tilde{x}^*_{\backslash G}$, then subtracting $p(\tilde{e}_{\backslash G} - \tilde{x}^*_{\backslash G}) = 0$ from both sides gives
$$p_G(\tilde{e}_G - \tilde{x}_G) = 0.$$
Since $p_G > 0$, the result follows. □

Definition 6.8 (Returns matrix). *If* q *is a vector of asset prices and* r *a matrix of asset returns, then the returns matrix of* $(-q, r)$ *is*
$$R = \begin{bmatrix} -q \\ r \end{bmatrix}.$$

Definition 6.9 (Financial budget correspondence). *The financial budget correspondence* $B : \mathbb{R}^{A,S+1} \times X^* \times X \rightrightarrows X \times \mathbb{R}^A$ *is*
$$B(R, p, e) = \{(x, z) \in X \times \mathbb{R}^A : p(e - x) = Rz\}.$$

Definition 6.10 (General financial equilibrium). *The triple* (x^*, z^*, q^*, P^*) *is a general financial equilibrium if*

- *Utility maximization:*
$$(x_h^*, z_h^*) \in \operatorname*{arg\,max}_{(x_h, z_h) \in B(R, P^*, e_h)} u_h(x_h).$$

- *Materials balance:* $\tilde{x}^* = \tilde{e}^*$.

- *Asset market clearing:* $\tilde{z}^* = 0$.

Lemma 6.11. *If* $u : X \to \mathbb{R}$ *is locally non-satiated spot-by-spot,*
$$(x^*, z^*) \in \operatorname*{arg\,max}_{x \in B^{\leq}(R, P, e)} u(x),$$
then $(x^*, z^*) \in B(R, P, e)$.

Proof. Suppose otherwise, that some spot has $p(s)x^*(s) < p(s)e(s) + Rz^*(s)$. Then there is an open ball containing $x^*(s)$ inside $B^<(R, P, e)(s)$, where $B^<$ is defined in the obvious way. Since u is locally non-satiated spot-by-spot, there must be some x' such that $(x', z^*) \in B^\leq(R, P, e)$ and $u(x') > u(x^*)$, violating the condition that (x^*, z^*) solved the utility maximization problem. □

Theorem 6.12. *If (x^*, z^*, p^*, q^*) is a weak GFE and every utility function u_h is locally non-satiated spot-by-spot, then it is a GFE.*

Proof. This is a trivial consequence of the previous lemma. □

Theorem 6.13 (Positive prices). *Suppose (x^*, z^*, p^*, q^*) is a weak GFE. If some househould's utility function u_h is increasing spot-by-spot with X_h unbounded above, then $p^*(s) > 0$ for each spot $s \in \mathcal{S}$.*

Proof. Without loss of generality, assume u_1 is increasing spot-by-spot with X_1 unbounded above.

First, it is clear that $p^*(s) \not\leq 0$ for all $s \in \mathcal{S}$. Now, suppose for the sake of contradiction that $p^{*c}(s) < 0$. Then partition C into $C^- = \{c \in C : p^{*c}(s) < 0\}$ and $C^+ = \{c \in C : p^{*c}(s) \geq 0\}$. Clearly both C^- and C^+ are non-empty. Set $p^- = \sum_{c \in C^-} p^{*c}(s)$ and $p^+ = \sum_{c \in C^+} p^{*c}(s)$. Finally, set $\hat{x}(\backslash s) = x^*(\backslash s)$ and $\hat{x}^c(s) = \frac{1}{|C^+|p^+}$ if $c \in C^+$ and $\hat{x}^c(s) = \frac{1}{|C^-|p^-}$ if $c \in C^-$. Then $p^*\hat{x} = p^*x^*$ and $\hat{x}(s) \gg x^*$ with $\hat{x} > x^*$, contradicting the assumption that x^* maximizes u_1 over $B(p^*, e_h)$. □

7 Production

Definition 7.1 (Technology function). *A technology function is a function $t : \mathbb{R}^G \to \mathbb{R}$. Its corresponding production set is $t^{-1}(\mathbb{R}_+)$.*

Definition 7.2 (Production shutdown). *A production set Y includes shutdown if $0 \in Y$.*

Definition 7.3 (Non-reversible production set). *Y is non-reversible if all $y \in Y$ have $-y \notin Y$.*

Definition 7.4 (No free lunch production set). *Y has no free lunch if every $y \in Y$ has $y \not> 0$.*

Definition 7.5 (Diminishing returns to scale). *Y has diminishing returns to scale if Y is convex.*

Definition 7.6 (Free disposal). *Y has free disposal if $y \in Y$ and $y' \leq y$ implies $y' \in Y$ also.*

Definition 7.7 (Production boundary condition). *Y satisfies the production boundary condition if there is some $\bar{y}$ such that $Y \ll \bar{y}$.*

Definition 7.8 (Aggregate production set). *The aggregate production set is $Y = \sum_{f \in \mathcal{F}} Y_f$.*

Even though production sets are of primary interest, it is convenient to reason about them with technology functions. Note that there are many technology functions that can represent the same production set. We will restrict our attention without (much) loss in generality to nice technology functions.

Proposition 7.9. *Each of the properties of a production set Y are related to the following conditions on its technology function t:*

- *Y includes production shutdown if $t(0) = 0$.*
- *Y has free disposal if t is decreasing.*
- *Y has diminishing returns to scale if t is quasiconcave.*
- *Y is irreversible if for all $y \in \mathbb{R}^G \backslash \{0\}$, $t(y) \geq 0$ implies $t(-y) < 0$.*
- *Y has no free lunch if $t(y) > 0$ implies $y \not> 0$.*

Definition 7.10 (Basic model with production). *(Primarily for existence)*
This is the basic model with the requirement

- *Production technology functions t_f are chosen such that sets Y_f are closed, include shutdown (with $t_f(0) = 0$), diminishing returns to scale (with t_f* concave*) and free disposal with t* strictly *increasing. The aggregate production set Y must be irreversible and have no free lunch.*

Definition 7.11 (Smooth model with production). *(For finite local uniqueness)*
This is the smooth model with the requirement

- *Technology functions $t_f : \mathbb{R}^G \to \mathbb{R}$ are C^2, differentiably strictly decreasing, differentiably strictly quasi-concave, each $t_f(0) = 0$ and satisfy the boundary condition. The aggregate production set Y is irreversible and has no free lunch.*

Definition 7.12 (General equilibrium with production). *The triple (x^*, y^*, p^*) of consumption, production and prices is a general equilibrium with production for some share allocation $s > 0$ with $\tilde{s} = \mathbf{1}$ if*

- *Utility maximization: Then we require*

$$x_h^* \in \underset{x_h \in B(p^*, p^*(e_h + s_h y^*))}{\arg\max} u_h(x_h).$$

- *Profit maximization:*

$$y_f^* \in \underset{y_f \in Y_f}{\arg\max}\ p^* y_f.$$

- *Market clearing: $\tilde{x}^* = \tilde{e}^* + \tilde{y}^*$.*

If $B(p^*, p^*(e_h + s_h y^*))$ is replaced with $B^{\leq}(p^*, p^*(e_h + s_h y^*))$, then (x^*, y^*, p^*) is a weak general equilibrium.

8 Complete Financial Markets

Definition 8.1 (No arbitrage). *A returns matrix constructed from $(-q, r)$ has no arbitrage if*

- *(Portfolio value version): there is no $z \in \mathbb{R}^A$ with $Rz > 0$.*
- *(Spot value version): there is some $\lambda \in \mathbb{R}^{S+1}_{++}$ such that $\lambda R = 0$.*

Note that the Portfolio value version can be equivalently stated as "there is no $z \in \mathbb{R}^A$ with $Rz < 0$". Intuitively, these are equivalent, since if I can surely lose money by buying an asset portfolio z, I can surely gain money by selling the same portfolio.

Theorem 8.2 (No arbitrage equivalence). *$(-q, r)$ has no arbitrage (porfolio version) if and only if it has no arbitrage (asset version).*

Proof. $\Leftarrow$: if $\lambda R = 0$, then $\lambda R z = 0$. But $\lambda \gg 0$, so $Rz \leq 0$.
$\Rightarrow$: Let $C^s = \text{cone}(\{r(s') : s' \in \{1, \ldots, S\} \backslash s\})$.

Step 1: No hyperplane separates $r(s)$ from $-C^s$. The portfolio version of NA requires that there is no $z \in \mathbb{R}^A$ with $Rz > 0$. In this case for every spot $s \in S$, there is no $z \in \mathbb{R}^A$ such that $r(s)z > 0$ and $R(\backslash s)z \geq 0$. Rewriting, there is no z such that $r(s) \in \text{int}(H^+(z))$ and $-C^s \subseteq H^-(z)$.

Step 2: $r(s) \in -C^s$ for all s. This follows from the previous step and Farkas lemma.

Step 3: There exists $(\lambda_{\backslash s}, 1)^T \in \mathbb{R}^{A-1}_+$ such that $(\lambda_{\backslash s}, 1)R = 0$. Since $r(s) \in -C^s$, there exists $\lambda_{\backslash s} \in \mathbb{R}^{A-1}_+$ such that $r(s) \cdot 1 = -\lambda_{\backslash s} R(\backslash s)$. This is clearly equivalent.

Step 4: There exists $\lambda \in \mathbb{R}^A_{++}$ such that $\lambda R = 0$. Put $\lambda = \sum_{s \in S} (\lambda_{\backslash s}, 1) \gg \mathbf{1}$. □

Definition 8.3 (No redundancy). *$(-q, r)$ has no redundancy if its returns matrix has full column rank.*

This is similar to Walras law for asset markets:

Theorem 8.4 (GFE market clearing redundancy). *Fix endowments e and returns r with no redudancy. Then (x^*, z^*, p^*, q^*) is a GFE if and only if (x^*, z^*, p^*, q^*) satisfies all the GFE conditions except $\tilde{z}^* = 0$.*

Proof. $\Longrightarrow$: trivial.

$\Longleftarrow$: Observe that $0 = P^*(\tilde{x}^* - \tilde{e}) = R\tilde{z}^*$, where the first equality come from the market clearing constraint $\tilde{x}^* = \tilde{e}$ and the second equality is the sum of the the households' budget constraints. But no redundancy requires $r\tilde{z}^* = 0$ only if $\tilde{z}^* = 0$. □

Theorem 8.5 (GFE implies no arbitrage). *If (x^*, z^*, p^*, q^*) is a GFE with endowments e and returns r, and some household's utility function u_h is increasing spot-by-spot, then $(-q, r)$ has no arbitrage.*

Proof. Suppose otherwise, that

$$x_h^* \in \operatorname*{arg\,max}_{(x_h, z_h) \in B_h(p^*, q^*)} u_h(x_h^*).$$

If there is arbitrage, then there is some $z' \in \mathbb{R}^a$ such that $Rz' > 0$. So some spot has $r(s)z' > 0$. Let $x_h'(s) = x^*(s) + r(s)z\frac{1}{C}(1/p^{1,s}, \ldots, 1/p^{c,s})$ for all spots s. Clearly some spot s has $x_h'(s) \gg x_h^*(s)$. Since u_h is increasing spot-by-spot, $u_h(x') > u_h(x^*)$. Moreover, $(x_h', z_h') \in B_h(p^*, q^*)$, contradicting the assumption that (x_h^*, q_h^*) solved household h's utility maximization problem. □

Definition 8.6 (Complete markets). *$(-q, r)$ is a complete market if r has full row rank.*

Definition 8.7 (Arrow securities). *A returns matrix $R = (-q, r)$ is constructed from Arrow securities if r is an identity matrix.*

Proposition 8.8. *Arrow securities $R = (-q, r)$ have complete markets and no redundancy. If $q \gg 0$, then it also has no arbitrage.*

Theorem 8.9 (Equivalence of Arrow securities). *Fix any securities $(-q, r)$ that have complete markets, no redundancy and no arbitrage. (x^*, p^*, z^*, q) is a GFE with securities $(-q, r)$ if and only if $(x^*, p^*, rz^*, qr^{-1})$ is a GFE with Arrow securities $(-qr^{-1}, I)$.*

Proof. Under the conditions on $(-q, r)$, r is an invertible square matrix, so the expressions in the theorem are well-defined.

The first budget constraint is equivalent as

$$\begin{aligned} & P^*(0)(x_h^*(0) - e_h(0)) = -qz_h^* \\ \iff & P^*(0)(x_h^*(0) - e_h(0)) = (-qr^{-1})(rz_h^*). \end{aligned}$$

The other budget constraints are equivalent, since

$$\begin{aligned} & P^*(\backslash 0)(x_h^*(\backslash 0) - e_h(\backslash 0)) = rz_h^* \\ \iff & P^*(\backslash 0)(x_h^*(\backslash 0) - e_h(\backslash 0)) = I(rz_h^*). \end{aligned}$$

Finally, the asset market clearing conditions are clearly equivalent with

$$\tilde{z^*} = 0 \iff r\tilde{z}^* = r0 = 0.$$

□

Theorem 8.10 (Arrow equivalence theorem). *Let $R = (-q, r)^T$ be a returns matrix with complete markets and no arbitrage and fix $x^* \in X$. There exists (z^*, p^*) such that (x^*, z^*, p^*, q) is a GFE if and only if there exists $\hat{p}$ such that $(x^*, \hat{p})$ is a GE.*

In particular, $\hat{p} = \lambda^* P^*$ and $p^*(s) = \frac{\hat{p}(s)}{\lambda^*(s)}$, where $(1, \lambda^*)$ is a NA value vector of R. If R has no redundancy, then $z^* = r^{-1}P^*(x - e)$.

Proof. From the previous theorem, we can assume without loss of generality that R is constructed from Arrow securities. In this case $\lambda^* = q$.

It suffices to show that the budget sets $B(P^*, R, e)$ and $B_h(\hat{p}, e)$ impose the same constraints on consumption with appropriate choices of p^* and $\hat{p}$. This is because the materials balance conditions in both equilibrium concepts are identical, and GFE's asset market clearing constraint is redundant by Theorem 8.4.

In the first direction, we suppose (x^*, z^*, p^*, q) is a GFE, and construct a GE with $\hat{p} = (1, \lambda^*)P^*$. In the reverse direction, we suppose $(x^*, \hat{p})$ is a GE and construct a GFE by putting $p^*(0) = \hat{p}(0)$ and $p^*(s) = \lambda^*(s)\hat{p}(s)$ for $s > 0$. In both cases, we have the identity $\hat{p} = (1, \lambda^*)P^*$.

If $x_h \in B(\hat{p}, e_h)$, then pick $z_h = P^*(\backslash 0)(x_h(\backslash 0) - e_h(\backslash 0))$. Then $P^*(\backslash 0)(x_h(\backslash 0) - e_h(\backslash 0)) = z_h = rz_h$ by construction. Moreover,

$$\begin{aligned}
\hat{p}(x_h - e_h) &= 0 \\
(1, \lambda^*)P^*(x_h - e_h) &= 0 \\
P^*(0)(x_h(0) - e_h(0)) &= -\lambda^* P^*(\backslash 0)(x_h(\backslash 0) - e_h(\backslash 0) \\
P^*(0)(x_h(0) - e_h(0)) &= -\lambda^* z_h \\
P^*(0)(x_h(0) - e_h(0)) &= -qz_h.
\end{aligned}$$

So $(x_h, z_h) \in B(P^*, R, e_h)$.

Conversely, if $(x_h, z_h) \in B(P^*, R, e_h)$, then $\hat{p}(x_h - e_h) = \lambda^* P^*(x_h - e_h) = \lambda^* R z_h = 0$, and $x_h \in B(\hat{p}, e_h)$. □

9 Optimality

Definition 9.1 (Feasible set). *Given resources $r \in \mathbb{R}^G$ and household consumption sets $X_h \subseteq \mathbb{R}^G_+$, the feasible set is*

$$X = \{x \in \prod_{h \in \mathcal{H}} X_h : \tilde{x} = r\}.$$

Definition 9.2 (Pareto set). *Given a feasible set X and utility functions $u_h : X_h \to \mathbb{R}$, the Pareto set is*

$$X^* = \{x \in X : \textit{there does not exist any } \hat{x} \in X \textit{ such that } u(\hat{x}) > u(x)\}.$$

Definition 9.3 (Utility possibility set). *Given a feasible set X and utility functions $u_h : X_h \to \mathbb{R}$, the utility possibility set U is*

$$U = \{u \in \mathbb{R}^H : \textit{there exists } x \in X \textit{ such that } u \le u(x)\}.$$

Definition 9.4 (Utility possibility frontier). *Given a feasible set X and utility functions $u_h : X_h \to \mathbb{R}$, the utility possibility frontier is $U^* = u(X^*)$.*

Definition 9.5 (Social welfare function). *A social welfare function is a function $f : \mathbb{R}^H \to \mathbb{R}$ that assigns a social welfare to each vector of household utilities.*

Lemma 9.6. *X is compact.*

Proof. First notice that $X = f^{-1}(\{r\})$, where function $f : \mathbb{R}^{GH} \to \mathbb{R}$ is the continuous function $f(x) = \tilde{x}$. So X is closed. X is clearly bounded with $X \subseteq B(0, \|r\|)$, so the Heine-Borel theorem implies X is compact. □

Lemma 9.7. *If each u_h is continuous, then U is closed. If each u_h is concave, then U is convex.*

Proof. If u_h is continuous, then u is also continuous and hence $u(X)$ is compact. Now suppose $a_k \to a$ is a convergent sequence in U. Then for each a_k, there is some $a_k^* \in u(X)$ such that $a_k \leq a_k^*$. Since $u(X)$ is compact, a_k^* has a convergent subsequence b_k^* such that $b_k^* \to b^* \in u(X)$. The corresponding subsequence of a_k, denoted b_k must have $b_k \to b = a$. Since each $b_k \leq b_k^*$, it follows that $a = b \leq b^*$, and hence $a \in U$. So U is closed.

If $a, b \in U$, then there must be some $x, y \in u(X)$ with $a \leq u(x)$ and $b \leq u(y)$. If $c \in [a, b]$, then it is clear from the concavity of u that there is some $z \in [x, y]$ with $c \leq u(z)$. So U is convex. □

Theorem 9.8 (Social welfare characterization of Pareto optimality). *Suppose each u_h is concave. If $x^* \in X^*$ then there is a social welfare function f_{x^*} that is linear and increasing in each household's utility such that*

$$x^* \in \arg\max_{x \in X} \; f_{x^*}(u(x)).$$

Conversely, if x^* maximizes a social welfare function f that is in addition strictly increasing in each household's utility, then $x^* \in X^*$.

Proof. $\Rightarrow$: Suppose $x^* \in X^*$. Then clearly $x^* \in \operatorname{bd} U$, which is convex by the previous lemma. So, by the supporting hyperplane theorem, there exists some hyperplane $H(p) - x^*$ such that $U \subseteq H^-(p) - x^*$. So let $f_{x^*}(u) = pu$.

$\Leftarrow$: If $x^* \in \arg\max_{x \in X} f(u(x))$, this means $f(u(x^*)) \geq f(u(x))$ for every $x \in X$. Since f is strictly increasing, this means some h has $u(x_h^*) \geq u(x_h)$, so x^* is Pareto optimal. □

Theorem 9.9 (First welfare theorem). *If (x^*, y^*, p^*) is a weak GE and u_h are locally non-satiated, then x^* is a Pareto optimal allocation.*

Proof. Suppose (x', y') is feasible and $u(x') > u(x^*)$. Since $u_h(x'_h) \geq u_h(x_h^*)$, the local non-satiation and the weak GE asumptions imply

$$p^*(x'_h - e_h - s_h y^*) \geq p^*(x_h^* - e_h - s_h y^*)$$

for all households. (By Theorem 6.12, (x^*, y^*, p^*) is a GE, and hence x_h^* is in household h's (strict) budget set. Local non-satiation then implies x'_h is not in the interior of the weak budget set.) Moreover, since $u(x') > u(x^*)$, this inequality is strict for some household. Summing up over households and firms gives

$$p^*(\tilde{x}' - \tilde{y}^*) > p^*(\tilde{x}^* - \tilde{y}^*).$$

Notice that by feasibility, $\tilde{x}^* = \tilde{y}^*$, so the right side is 0. Together with the feasibility assumption that $\tilde{x}' = \tilde{y}'$, this implies

$$\begin{aligned} & p^*(\tilde{x}' - \tilde{y}^*) > 0 \\ \Longrightarrow \quad & p^*\tilde{y}' > p^*\tilde{y}^*. \end{aligned}$$

This final inequality clearly implies that some firm f has a suboptimal production plan y_f^* given prices p^*, contradicting the assumption that (x^*, y^*, p^*) as a weak GE. □

Counter-examples:

- local non-satiation: equilibrium can be in the interior of one household's indifference curve.
- missing markets.

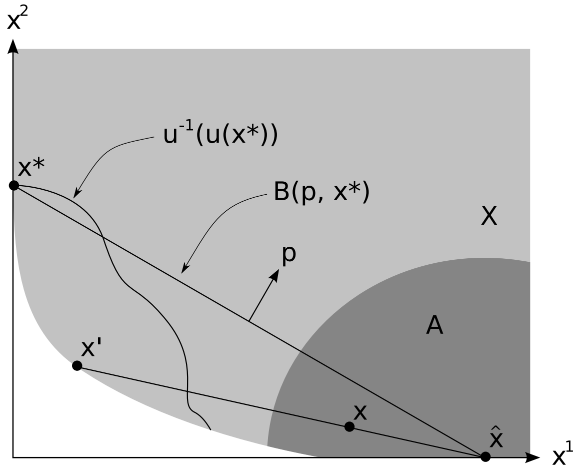

Figure 6: If x^* does not solve utility max, then it does not solve expend min. That is, if some affordable $\hat{x}$ gives higher utility, then we can construct some x that is cheaper than x^*.

Theorem 9.10 (Duality). *Consider the following dual propositions:*

$$\text{(expend min)} \;\; x^* \in \underset{x \in U(x^*)}{\arg\min} \; px,$$
$$\text{(utility max)} \;\; x^* \in \underset{x \in B(p,x^*)}{\arg\max} \; u(x).$$

If $X \subseteq \mathbb{R}^n_+$ is convex, $x^* \in X$, there is some $x' \in X$ with $px' < px^*$, and the utility function $u : X \to \mathbb{R}$ is C^0 then (expend min) $\implies$ (utility max).

Conversely, if X is convex, unbounded above and, and the utility function $u : X \to \mathbb{R}$ is C^0 and increasing then (utility max) $\implies$ (expend min).

Proof. $\implies$: This part of the proof is illustrated in Figure 6. Firstly, $x^* \in B(p, x^*)$ holds trivially. So if $x^* \notin \arg\max_{x \in B(p,x^*)} u(x)$, it must be because some $\hat{x} \in B(p, x^*)$ has $u(\hat{x}) > u(x^*)$.

By continuity of u, there is an open neighbourhood $A \subseteq X$ of $\hat{x}$ with $u(A) > u(x^*)$. Also $(x', \hat{x}) \subseteq X$ by the convexity of X. Pick any $x \in (x', \hat{x}) \cap A$ which is clearly non-empty. Firstly, $u(x) > u(x^*)$ since $x \in A$. Secondly, $px < px^*$ by the linearity of $x \mapsto px$. More specifically, since $x = \alpha x' + (1 - \alpha)\hat{x}$, we can write

$$px = \alpha px' + (1 - \alpha)p\hat{x} < \alpha p\hat{x} + (1 - \alpha)p\hat{x} = p\hat{x} = px^*.$$

$\impliedby$: Suppose for the sake of contradiction that some $\hat{x} \in U(x^*)$ has $p\hat{x} < px^*$. Then let

$$x' = x + p(x^* - \hat{x})/n(1/p^1, \ldots, 1/p^G),$$

which clearly has $px' = px^*$. Also $x' \gg x$, and since u is increasing $u(x') > u(x)$, contradicting the assumption that x^* is utility maximizing over $B(p, x^*)$. □

The second welfare theorem gives conditions under which a Pareto optimal allocation can be supported in a general equilibrium.

Theorem 9.11 (Second welfare theorem). *Suppose $x^* \in \text{int}(X)$ is Pareto optimal given endowments $\tilde{e}$ and*

- $X_h \subseteq \mathbb{R}^G_+$ *is convex.*

- *u_h is C^0 and quasiconcave.*
- *X_1 is unbounded above and u_1 is increasing.*

Then there exists $\tau^* \in \mathbb{R}^H$ and $p^* \in \mathbb{R}^G_+$ such that (x^*, p^*, τ^*) is a weak GE with transfers.

Proof. The first five steps of this proof establish prices under which the allocation x^* are solutions to the households' expenditure minimization problem. This situation is sometimes called a quasi-equilibrium. The final two steps establish that the quasi-equilibrium is also a general equilibrium using the duality theorem.

Let $Z = \sum_h U_h(x_h^*)$ be the set of resources which support allocations that weakly Pareto improve x^*.

Step 1: Z is convex: Since each u_h is quasiconcave, each $U_h(x_h^*)$ is convex and Z is convex.

Step 2: $\tilde{x}^* \in \operatorname{bd} Z$: Since $x^* \in U(x^*)$, it is clear that $\tilde{x}^* \in Z$. Less obvious is that $\tilde{x}^* \in \operatorname{bd} Z$. For each ϵ, define $x'(\epsilon)$ to be the allocation with $x'_1(\epsilon) = x_1^* - \epsilon\mathbf{1}$, and $x'_h(\epsilon) = x_h^*$ for $h \neq 1$.

Suppose for the sake of contradiction that $\tilde{x} - \epsilon\mathbf{1} \in Z$. Note that for sufficiently small ϵ, this allocation is inside X, since $x^* \in \operatorname{int}(X)$. Then there is some allocation x with $u(x) \geq u(x^*)$ and $\tilde{x} = \tilde{x}^* - \epsilon\mathbf{1}$. But setting $x' = (x_1 - \epsilon\mathbf{1}, x_{\setminus 1})$ gives an allocation that Pareto dominates x^*, violating a condition of the theorem.

So, we can construct a sequence outside Z converging to x^* by letting $\epsilon \to 0$. So $x^* \in \operatorname{bd} Z$.

Step 3: Construct p^* with $Z \subseteq H^+(p^*) + \tilde{x}^*$: The supporting hyperplane theorem now implies that there exists some hyperplane $H(p^*) + \tilde{x}^*$ such that $Z \subseteq H^+(p^*) + \tilde{x}^*$. In algebraic terms, this means $p^*(z - \tilde{x}^*) \geq 0$, or equivalently, $p^* z \geq p^* \tilde{x}^*$ for all $z \in Z$.

Step 4: $p^* > 0$: The supporting hyperplane theorem in the previous steep guaranteed $p^* \neq 0$. Suppose for the sake of contradiction that some $p^{*g} < 0$. Then set $x = (x_1^* + 1^g, x_{\setminus 1}^*)$, where 1^g is a vector of zeros, except for a one in the g^{th} position. Then $\tilde{x} \in Z$ since $u(x) \geq u(x^*)$. But $p^*\tilde{x} < p^*\tilde{x}^*$, violating the conclusion from the previous step.

Step 5: x_h^* solves h's expend min problem: Now, consider any $\hat{x}_h \in U_h(x_h^*)$. Then $\tilde{x}^* + (\hat{x}_h - x_h^*) \in Z$, and hence $p^*[\tilde{x}^* + (\hat{x}_h - x_h^*) - \tilde{x}^*] = p^*(\hat{x}_h - x_h^*) \geq 0$. This means

$$x_h^* \in \operatorname*{arg\,min}_{x_h \in U_h(x_h^*)} p^* x_h.$$

Step 6: x_h^* solves h's utility max problem: Since $x_h^* \in \operatorname{int}(X_h)$ and $p^* > 0$, there is some $x'_h \in X$ with $px'_h < px_h^*$ and the duality theorem then implies x_h^* solves h's utility max problem.

Step 7: Equilibrium with transfers: Finally, putting $\tau_h^* = p^*(x_h^* - e_h)$ gives an equilibrium with transfers τ^*. This implies $p^* x_h^* = p^* e_h + \tau_h^*$, which gives the correct budget constraint. Moreover, since x^* is feasible, $\tilde{\tau}^* = p^*(\tilde{x}^* - \tilde{e}) = p^* 0 = 0$. □

Counter-examples:

- interior: Arrow's counterexample. If $x^* \in \operatorname{bd} X$, then there might not be a cheaper point, and the duality theorem no longer applies. In Arrow's counterexample, there are two households with two commodities, one household with strictly increasing utility in both goods, and the other indifferent about good 1. In this example, no price for good 1 can support x^*. If the $p^1 = 0$, then the first household can attain an arbitrarily high utility, so there is no maximizer. If you cook the first household's utility function the right way, you can rule out $p^1 > 0$ by making the derivative get infinitely steep close to x^*.

10 Existence

Theorem 10.1 (Weak budget correspondence properties). *Suppose $X \subseteq \mathbb{R}^G_+$ is compact. Let $B_e^{\leq} : \Delta \rightrightarrows X$ be the weak budget correspondence*

$$B_e^{\leq}(p) = \{x \in X : p(e - x) \geq 0\},$$

and assume $e \gg 0$.

1. *$B_e^{\leq}$ is non-empty, convex-valued and compact-valued for all $e \in X$.*
2. *$B_e^{\leq}$ is continuous for all $e \in X$.*

Proof. 1. $B_e^{\leq}$ is trivially non-empty valued, as $e \in B_e^{\leq}(p)$.

Let $f_{p,e}(x) = p(e-x)$. Notice that $B_e^{\leq}(p) = f_{p,e}^{-1}(\mathbb{R}^+)$. Since f is continuous in x, and $\mathbb{R}^+$, we can conclude that $B^{\leq}(p)_e$ is closed in X. Since X is closed, it is also closed in $\mathbb{R}^n$. Finally, every closed subset of a compact set is compact.

Finally, $x \mapsto p(e-x)$ is affine, so $B_e^{\leq}$ is clearly convex valued.

2. **Upper hemi-continuous**: By Proposition 3.7 it suffices to show that $B_e^{\leq}$ has the upper sequence property; namely every sequence $(p_n, x_n) \in \text{graph}(B_e^{\leq})$ with $p_n \to p$ has a convergent subsequence with limit $(p, x) \in \text{graph}(B_e^{\leq})$.

Since $\Delta \times X$ is compact, (p_n, x_n) has a convergent subsequence, which we can assume without loss of generality is itself. Let $f_e(p,x) = p(e-x)$. Notice that $\text{graph}(B_e^{\leq}) = f_e^{-1}(\mathbb{R}_+)$. Clearly, f_e is continuous, so $(p_n, x_n) \to (p, x)$ implies $f_e(p_n, x_n) \to f_e(p, x)$. In particular, each $f_e(p_n, x_n) \geq 0$, so $f_e(p,x) \geq 0$ and hence $(p,x) \in \text{graph}(B_e^{\leq})$.

Lower hemi-continuous: Suppose $p_n \to p$ and $x \in B_e^{\leq}(p)$. It suffices to show that there is some $x_n \in B_e^{\leq}(p_n)$ with $x_n \to x$.

Let $x_n = \alpha_n x$, where

$$\alpha_n = \begin{cases} \dfrac{pxp_ne}{p_nxpe} & \text{if } p_nx > 0, \\ 1 & \text{if } p_nx = 0. \end{cases}$$

Note that $pe > 0$ since $e \gg 0$, so α_n is well-defined. Clearly, $\alpha_n \to 1$, so $x_n \to x$. Moreover,

$$\begin{aligned} & p_n(e - x_n) \geq 0 \\ \iff & p_n\left(e - \frac{pxp_ne}{pe}\right) \geq 0 \\ \iff & p(e-x) \geq 0. \end{aligned}$$

So, $x_n \in B_e^{\leq}(p_n)$ as required.

□

Lemma 10.2 (Constraint relaxation lemma). *Suppose $u : X \to \mathbb{R}$ is continuous, quasiconcave and locally non-satiated, where X is convex. Let $\bar{X} \subseteq X$. If*

$$x^* \in \underset{x \in B^{\leq}(p,e) \cap \text{int}(\bar{X})}{\arg\max} u(x)$$

then

$$x^* \in \underset{x \in B^{\leq}(p,e)}{\arg\max}\, u(x).$$

Proof. Since u is locally non-satiated, any maximizer x^* must lie in $B(p,e)$. Now, suppose for the sake of contradiction that some $\hat{x} \in B(p,e)$ has $u(\hat{x}) > u(x^*)$. By continuity, there must be an open set $N \subset X$ containing $\hat{x}$ with $u(N) > u(x^*)$. In particular $u(\hat{x} - \epsilon\mathbf{1}) > u(x^*)$ for some $\epsilon > 0$. By quasiconcavity, $u((x^*, \hat{x}) \cap B^{\leq}(p,e)) \geq u(x^*)$. But if the constraint is not binding (with $x^* \in \text{int}(\bar{X})$), some of the points in $(x^*, \hat{x}) \cap B^{\leq}(p,e)$ are in the interior of $B^{\leq}(p,e)$, violating local non-satiation. □

Theorem 10.3 (Existence of weak GE). *In the basic model with u_1 increasing, a weak GE (x^*, p^*) exists.*

Proof. From the previous theorem, the budget set correspondences $B_h^{\leq}$ are continuous, non-empty, and convex- and compact-valued. Let $\bar{X}_h$ be the intersection of X with the compact cube with corners 0 and $\tilde{e} + \mathbf{1}$. Let $\bar{X} = \prod_h \bar{X}_h$. Clearly, $B_h^{\leq} \bar{X}_h$ retains all of these properties. Since every equilibrium must have $\tilde{e} = \tilde{x}$, it suffices to show that an equilibrium exists in $\bar{X}$. Clearly, this extra compactification constraint can't be binding, so by continuity, quasiconcavity and local non-satiation, the constraint relaxation lemma applies, and the constraint can be relaxed.

Since u is continuous, so the demand correspondence

$$D_h(p, e_h) = \arg\max_{x'_h \in B_h^{\leq} \cap \bar{X}} \; u_h(x'_h)$$

is UHC by Berge's maximum theorem. It is also clearly non-empty and convex-valued (since u_h is quasiconcave).

Let the price correspondence $P : \bar{X} \rightrightarrows \Delta^G$ be the function

$$P(x) = \arg\max_{p \in \Delta^G} \; p(\tilde{x} - \tilde{e})$$

that picks prices that maximize the value of excess demand. Clearly, the objective function is continuous, and the constraint is continuous (as it is constant), convex, compact and non-empty. So Berge's maximum theorem again gives P is UHC.

Let $f : \Delta^G \times \bar{X} \rightrightarrows \Delta^G \times \bar{X}$ be the self-map

$$f(p, x) = \{(p', x') : p' \in P(x), x'_h \in D_h(p, e_h)\}.$$

We will first show that f has a fixed point, and then show how to construct a weak GE from any fixed point.

First, f is clearly UHC, non-empty and convex valued, and is a self-map on a convex and compact set. So Kakutani gives us a fixed point.

Second, suppose (p^*, x^*) is a fixed point. This means $x_h^* \in x_h(p^*)$, so the allocation solves the households' utility maximization problems. However, x^* may not have aggregate consistency – namely $\tilde{x}^* = r$. I will show that $\tilde{x}^* \leq r$. Then, since u_1 is increasing, we can give any excess to them. This will not violate the budget constraint, since prices will be 0 for goods with this excess.

Now, $x_h^* \in D_h(p^*, e_h)$ which means $p^*(x_h^* - e_h) \leq 0$ and hence $p^*(\tilde{x}^* - r) \leq 0$. Local non-satiation now implies $p^*(x_h^* - e_h) = 0$. Summing up implies $p^*(\tilde{x^*} - r) = 0$. So if $\tilde{x}^{*g} < r^g$, then $p^{*g} = 0$.

Since $p^* \in \Delta^G$, we have $p^* > 0$. Notice that from the definition of $P(x)$, if some $\tilde{x}^{*g} - r^g > 0$, then setting $p^{*g} = 1$ would maximize $p^*(\tilde{x}^* - r)$, leaving it greater than 0. So we can conclude that $\tilde{x}^* - \tilde{e} \leq 0$. □

Counter-examples:

- interior: if $e_h^g = 0$ for all but one household, and the household with the endowment is the only household that wants it, then no price will give an eq.

11 Determinacy

Recall the smooth analysis framework is made up of variables, parameters and equations. The pairs of variables and parameters that solve the equations are solutions. Smooth analysis studies how the solutions change when the parameters change. In the general equilibrium model, these components are:

- **Variables.** (sometimes called *endogeneous variables* or *dependent variables* in the olden days.) These are allocations $x_h \in X_h$, Lagrange multipliers $\lambda_h \in \mathbb{R}$ and prices[2] $p^{\setminus G} \in \mathbb{R}^{G-1}$. We will put these into a $H(G+1) + G - 1$ dimensional vector, $(((x_h, \lambda_h)_{h \in \mathcal{H}}), p^{\setminus G})$.

[2]Without loss of generality, we can set $p^G = 1$, and drop p^G from our analysis. Then we write $p = (p^{\setminus G}, 1)$.

- **Parameters.** (sometimes called *exogenous variables* or *independent variables* in the olden days.) These are endowments $e_h \in X_h$. This can be written as a HG-dimensional vector $e \in X$.

- **Equations.** The solutions of these equations form the set of general equilibria:

$$\begin{bmatrix} (Du_h(x_h) - \lambda_h(p^{\setminus G}, 1))^T \\ (p^{\setminus G}, 1)(e_h - x_h) \\ \sum_h e_h^{\setminus G} - \sum_h x_h^{\setminus G} \end{bmatrix} = 0.$$

  These equations are sometimes called the *extended form equations.* The vectors on the left and right are $HG + H + (G - 1)$ dimensional. The first HG equations are the first order conditions for each household's utility maximization problem. The next H equations are the budget constraints. The final $G - 1$ equations are the market clearing constraints. Recall that Walras law allows us to drop one of them without loss of generality.

Table 1 relates these variables, parameters and equations, and the spaces they lie in to the smooth analysis framework.

Math	Application	Description
J	$H(G+1) + G - 1$	number of variables
ξ	$((x_h, \lambda_h)_{h \in \mathcal{H}}, p)$	a variable
Ξ	$\left(\prod_{h=1}^{H} X_h \times \Lambda_h\right) \times P^{G-1}$	variable space
K	GH	number of parameters
θ	e	a parameter
Θ	$E = \mathbb{R}^{HG}_{++}$	parameter space
L	$H(G+1) + G - 1$	number of equations
$\Phi : \Xi \times \Theta \to \mathbb{R}^L$	left side of equations above	the smooth function
$M = \Phi^{-1}(0)$	$((x_h, \lambda_h)_{h \in \mathcal{H}}, p^{\setminus G}, e_h)$ that are Walrasian equilibria	solution set (manifold)

Table 1: Application of Smooth Analysis to General Equilibrium

Theorem 11.1 (Smooth characterization of GE). *The following smooth equation characterizes general equilibria in the smooth model.*

$$\Phi(x, p^{\setminus G}, \lambda, e) = \begin{bmatrix} (Du_h(x) - \lambda_h(p^{\setminus G}, 1))^T \\ (p^{\setminus G}, 1)(e_h - x_h) \\ \tilde{e}^{\setminus G} - \tilde{x}^{\setminus G} \end{bmatrix} = 0.$$

That is, $(x^*, (p^{*\setminus G}, 1))$ is a general equilibrium for endowments e if and only if there exists some λ^* such that $\Phi(x^*, e, (p^{*\setminus G}, 1), \lambda^*) = 0$.

Proof. This is mostly a straightforward application of the Lagrange necessary and sufficient conditions. Recall that a general equilibrium requires households to be maximizing subject to budget constraints, and materials to be balanced.

I will write $p^* = (p^{*\setminus G}, 1)$.

Utility max implies first two equations: The utility maximization requirement for household h is

$$x_h^* \in \arg\max_{x_h \in g_h^{-1}(0)} u_h(x_h),$$

where $g_h(x_h) = p^*(x_h - e_h)$, which is the second component of Φ. Since u is differentiable and $Dg = p^* \neq 0$ has full rank, the Lagrange theorem implies that any x_h^* that solves the utility maximization problem must have some $\lambda_h^* > 0$ such that $Du_h(x_h^*) - \lambda_h^* Dg_h(x_h^*) = 0$, which is the first component of Φ.

First two equations imply utility max: If $Du_h(x_h^*) - \lambda_h^* Dg_h(x_h^*) = 0$ and $g_h(x_h^*) = 0$, then since u is quasiconcave and $Du(x_h^*) \neq 0$ (since u is differentiably strictly increasing), the sufficient conditions imply x_h^* solves the utility maximization problem.

Materials balance: The materials balance constraint is equivalent to the equality in the component above by Walras law. □

Theorem 11.2 (Properness in smooth model). *In the smooth equilibrium model, the manifold projection $\pi : M \to \Theta$ of $M = \Phi^{-1}(0)$ is proper.*

Proof. We need to show π is continuous and for all compact subsets $\bar{E} \subseteq E$ of the endowment space, $\pi^{-1}(\bar{E})$ is also compact. Or, in the language of smooth analysis, we need to show for all compact subsets $\bar{\Theta} \subseteq \Theta$ of the parameter space, $\pi^{-1}(\Theta)$ is also compact.

π is a projection, and continuity of projections is an elementary result. (Sketch: projection is linear, and every linear function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous.)

Now pick any compact $\bar{E} \subseteq E$. We will show that $\pi^{-1}(\bar{E})$ is compact using the sequential characterization of compactness. That is, we will show that every sequence $\eta^\nu = (\xi^\nu, \theta^\nu) = (((x_h^\nu, \lambda_h^\nu)_{h\in\mathcal{H}}, p^\nu), e^\nu) \in \pi^{-1}(\bar{E})$ has a convergent subsequence. The remainder of the proof is organized into the following steps:

1. There is a subsequence of η^ν (without loss of generality assume this subsequence is η^ν) in which $e^\nu \to e \in \bar{E}$.
2. There is a subsequence of η^ν (WLOG itself) in which $x^\nu \to x \in X$.
3. There is a subsequence of η^ν (WLOG itself) in which $\lambda^\nu \to \lambda \in \Lambda$ and $(p^{\setminus G})^\nu \to p^{\setminus G} \in \mathcal{P}$.
4. The previous steps showed that there is a subsequence of η^ν (WLOG itself) in which $\eta^\nu \to \eta$. The final step shows $\eta \in M$ to conclude $\eta \in \pi^{-1}(\bar{E})$.

Step 1: There is a subsequence (WLOG itself) in which $e^\nu \to e \in \bar{E}$

Clearly $\pi^{-1}(\bar{E}) \subseteq \Xi \times \bar{E}$. So e^ν is a subsequence of the compact set $\bar{E}$, and must have a convergent subsequence.

Step 2: There is a subsequence (WLOG itself) in which $x^\nu \to x \in X$

First notice that each x_h^ν is bounded from below by 0 and above by $\bar{e} = \sup_n \tilde{e}^\nu$ (where the sup is coordinate-wise). So x_h^ν lies in a compact set and has a convergent subsequence, WLOG itself.

Next, since $e_h^\nu \to e_h \in X_h$, there is some $\epsilon > 0$ such that there is some convergent subsequence with $x_h^\nu \gg e_h - \epsilon\mathbf{1}$, where $\mathbf{1}$ is a vector of ones. So the sequence x_h^ν lies inside the closed set $\mathrm{cl}(U_h(e_h - \epsilon\mathbf{1}))$, so its limit x_h must also. By the boundary property of u_h, we know $\mathrm{cl}(U_h(\cdot)) \subseteq X_h$, so we conclude $x_h \in X_h$.

Step 3: There is a subsequence (WLOG itself) in which $\lambda^\nu \to \lambda \in \Lambda$ and $(p^{\setminus G})^\nu \to p^{\setminus G} \in \mathcal{P}$

Again, pick any household h. From the first order condition (i.e. the first part of the $\Phi(\eta) = 0$ equality),

$$Du_h(x_h^\nu) = {\lambda_h}^\nu p^\nu = \lambda_h^\nu((p^{\setminus G})^\nu, 1).$$

The last coordinate of this equality is

$$D_{x^G} u_h(x_h^\nu) = \lambda_h^\nu.$$

Now by step 2 and the smooth model assumptions on u_h,

$$\lim_{\nu\to\infty} \lambda_h^\nu = \lim_{\nu\to\infty} D_{x^G} u_h(x_h^\nu) = D_{x^G} u_h(x_h) \gg 0.$$

So $\lambda_h^\nu \to \lambda_h \in \Lambda_h$. The remaining coordinates in the first order condition now simplify to

$$\frac{D_{x^{\setminus G}} u_h(x_h^\nu)}{D_{x^G} u_h(x_h^\nu)} = p^{\setminus G\nu}.$$

So $p^{\setminus G\nu} \to p^{\setminus G} \gg 0$.

Step 4: $\eta \in M$

Since $\Phi(\eta^\nu) = 0$ and Φ is C^0,

$$\lim_{\nu\to\infty} \Phi(\eta^\nu) = \Phi\left(\lim_{\nu\to\infty} \eta^\nu\right) = \Phi(\eta) = 0.$$

So $\eta \in M$ as required. □

Theorem 11.3. *In the smooth model,* $\operatorname{rank}(D\Phi(x,p,\lambda,e)) = L$ *for all* $(x,p,\lambda,e) \in M = \pi^{-1}(0)$.

Proof. Recall the definition of Φ is

$$\Phi(x, p^{\setminus G}, \lambda, e) = \begin{bmatrix} (Du_h(x) - \lambda_h(p^{\setminus G},1))^T \\ (p^{\setminus G},1)(e_h - x_h) \\ \tilde{e}^{\setminus G} - \tilde{x}^{\setminus G} \end{bmatrix}.$$

The derivative of Φ is

$$D\Phi(x,p,\lambda,e) = \begin{bmatrix} D^2u_1(_1) & 0 & \begin{bmatrix}-\lambda_1 I \\ 0\end{bmatrix} & -(p^{\setminus G},1)^T & 0 & 0 & \\ 0 & \ddots & \vdots & 0 & \ddots & & \ddots \\ -(p^{\setminus G},1) & 0 & (e_1^{\setminus G} - p^{\setminus G})^T & 0 & & (p^{\setminus G},1) & 0 \\ 0 & \ddots & \vdots & & \ddots & 0 & \ddots \\ [-I\ \ 0] & \dots & 0 & 0 & \dots & [I\ \ 0] & \dots \end{bmatrix}.$$

Notice that it is organized into 3×4 blocks, where the 3 rows correspond to the different parts of the function (first order condition, budget constraint and materials balane), and the 4 columns are the derivatives with respect to x, p, λ and e respectively.

To show that $D\Phi$ has full rank, it suffices to show it has full row rank; that is, for every row vector $(\Delta x, \Delta\lambda, \Delta p)$,

$$(\Delta x, \Delta\lambda, \Delta p) D\Phi(x,p,\lambda,e) = 0 \implies (\Delta x, \Delta\lambda, \Delta p) = 0.$$

(Note that the choice of notation for the row vector is arbitrary.)

So, suppose $(\Delta x, \Delta\lambda, \Delta p) D\Phi(x,p,\lambda,e) = 0$.

1. The e_h^G columns of the equation are $\Delta\lambda_h e_h^G = 0$, which implies $\Delta\lambda_h = 0$.
2. The $e_h^{\setminus G}$ column block of the equation are $\Delta\lambda_h p^{\setminus G} + \Delta p^{\setminus G} = 0$. Since $\Delta\lambda_h = 0$, this reduces to $\Delta p^{\setminus G} = 0$.
3. The λ_h columns of the equation are $\Delta x_h (p^{\setminus G},1)^T = 0$.
4. Finally, the x_h columns give
$$\Delta x_h D^2 u_h(x_h) - \Delta\lambda_h (p^{\setminus G},1) = 0.$$
Multiplying on the right by $(\Delta x_h)^T$ gives
$$\Delta x_h D^2 u_h(x_h)(\Delta x_h)^T - \Delta\lambda_h (p^{\setminus G},1)(\Delta x_h)^T = 0.$$
From the previous step, the last term is 0, so the equation becomes
$$\Delta x_h D^2 u_h(x_h)(\Delta x_h)^T = 0.$$
Recall that u_h is differentiably strictly quasiconcave, which means that if $\Delta x_h \neq 0$ and $Du_h(x_h)(\Delta x_h)^T = 0$, then $\Delta x_h D^2 u_h(x_h)(\Delta x_h)^T < 0$. Moreover, u_h is differentiably strictly increasing, so $Du_h(x_h) \gg 0$. So the only possibility is that $\Delta x_h = 0$.

□

Theorem 11.4 (Finite local uniqueness of GE). *In the smooth model, a generic set of endowments, namely Θ^r has a finite locally unique set of equilibria.*

Proof. Firstly, Theorem 11.2 asserts π is proper, which means the stack of records theorem implies that π^{-1} is a finite locally unique correspondence on Θ^r.

Secondly, Theorem 11.3 implies $\text{rank}(D\Phi) = L$ everywhere, so the transversality theorem implies Θ^c has zero measure. Since π is proper, the closedness theorem to prove that Θ^c is closed. Together these establish that Θ^r is generic. □

Theorem 11.5 (Local uniqueness of GE). *In the smooth model, all endowments in an open set containing the Pareto set have a unique equilibrium.*

Proof. Pick any Pareto optimal endowment $e \in X^*$. The second welfare theorem implies there exists some prices $(p^{*\setminus G}, 1)$ such that $((p^*_{\setminus G}, 1), x^*)$ is an equilibrium.

Suppose there is a different equilibrium $((\hat{p}^{\setminus G}, 1), \hat{x})$. Note that if $\hat{x} = x^*$, then the first order condition

$$Du_h(x^*_h) + \lambda(p^*_{\setminus G}, 1) = 0$$

implies that $\hat{p}^{\setminus G} = p^{*\setminus G}$.

So, we must have $\hat{x} \neq x^*$. First, note that $u_h(\hat{x}_h) \geq u_h(x^*_h)$ for all households $h \in \mathcal{H}$, since $e_h \in B(\hat{p}, e_h)$. Since x^* is Pareto optimal, there can be no strict inequality. So the only remaining case is $u_h(\hat{x}_h) = u_h(x^*_h)$ with $\hat{x}_h \neq x^*_h$. But this violates strict quasi-concavity of u, as any point in $(\hat{x}_h, x^*_h)$ is in the convex budget set $B(\hat{p}, e_h)$.

Then show all $e \in X^*$ are a regular values, and then apply the finite local uniqueness theorem. This delivers an open set around each e which has finite local uniqueness, and hence local uniqueness. The union of all these open sets is open. □

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