Finding Euler Equations with the Envelope Theorem

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The Euler equation (EE) of a dynamic programming problem (DPP) gives the relationship between the payoffs in one period and the next in an optimal solution. In a sequential formulation of a DPP, the EE is easy to find with standard calculus techniques. However, recursive formulations of DPPs starting with a Bellman equation are often easier to work with. In this case, simple calculus does not lead easily to the EE. Fortunately, the envelope theorem provides a simple technique. I will state and prove a simple version of the envelope theorem, and then show how to use it to obtain the EE.

Firstly, the envelope theorem says that the derivative of the value function of a parametric optimization problem only depends on the parameter's direct effect on the objective, and not through any indirect effect on the optimal choice. I will show later how this lack of indirect effect simplifies matters.

Theorem 1 (Envelope Theorem). *Let* $f : X \times \Theta \to \mathbb{R}$ *and let* $g : \Theta \to X$ *be a function such that*

$$g(\theta) \in \underset{x \in X}{\arg\max}\ f(x; \theta)$$

for all $\theta \in \Theta$*. Finally, let* $V : \Theta \to \mathbb{R}$ *be the value function defined by* $V(\theta) = f(g(\theta); \theta)$*. If* f *and* g *are differentiable, then* V *is differentiable with*

$$DV(\theta) = D_2 f(g(\theta); \theta).$$

Proof. By the chain rule, for all $\theta \in \Theta$,

$$\begin{aligned} DV(\theta) &= Df(g(\theta); \theta)[Dg(\theta), 1]^T \\ &= D_1 f(g(\theta); \theta) Dg(\theta) + D_2 f(g(\theta), \theta). \end{aligned}$$

But $g(\theta)$ is a maximum of $f(\cdot; \theta)$, so $D_1 f(g(\theta), \theta) = 0$. This leaves $DV(\theta) = D_2 f(g(\theta), \theta)$. □

Now consider the Bellman equation of a household's investment and consumption dynamic programming problem

$$\begin{gathered} V(a) = \max_{c, a'} u(c) + \beta V(a'), \\ \text{s.t. } c + a' = F(a). \end{gathered}$$

Here a and c represents today's assets and consumption respectively. The $'$ decoration represents tommorrow's variables, so that c' is tommorrow's consumption. $F(a)$ is the total resources available after investing a.

The envelope theorem provides a simple technique for differentiating V, and hence finding a relationship between consumption and investment in present and subsequent time periods.

I will say that (c, a') is feasible with respect to (wrt) a if $c + a' = F(a)$.

Proposition 1 (Euler equation). *Suppose u, V and F in the above DPP are differentiable. Fix any a. If*

- *(c, a') is feasible wrt a, attaining $V(a) = u(c) + \beta V(a')$, and*
- *(c', a'') is feasible wrt a', attaining $V(a') = u(c') + \beta V(a'')$*

then $Du(c) = \beta Du(c')DF(a)$.

Proof. V can be collapsed into an unconstrainted optimization problem

$$V(a) = \max_{a'} u(F(a) - a') + \beta V(a').$$

To apply the envelope theorem, we need to decompose $V(a)$ into an objective function $f(a'; a)$ and an optimal solution function $g(a)$:

$$\begin{aligned} f(a'; a) &= u(F(a) - a') + \beta V(a') \\ g(a) &= \arg\max_{a'}\ f(a'; a). \end{aligned}$$

Now V can be written as $V(a) = f(g(a), a)$. The envelope theorem asserts that $DV(a) = D_2 f(g(a); a)$ for all a, which is

$$DV(a) = D_2 f(g(a); a) = Du(F(a) - g(a))DF(a).$$

This allows us to compute the first-order condition of the optimization problem for $g(a)$:

$$\begin{aligned} 0 &= D_1 f(g(a); a) \\ 0 &= Du(F(a) - g(a))(-1) + \beta DV(g(a)) \\ Du(F(a) - g(a)) &= \beta Du(F(g(a)) - g(g(a)))DF(a). \end{aligned}$$

Rewriting this in the original notation, this equation reads $Du(c) = \beta Du(c')DF(a)$. $\square$