ECON 897. Final/Waiver Exam. Tuesday, September 06, 2005

You have 150 minutes (two hours and a half). The exam has 150 points and consists of three parts. Each part has a total of 50 points.

Read carefully and think before you write. Good luck!

Part I (50 points)

Q1 (10 points). **Warming up questions**

(1.1) Show that if a sequence is Cauchy and has a convergent subsequence, it is also convergent.

(1.2) Let X be a set and $d : X \times X \to \mathbb{R}$ a distance defined as: (i) $d(x, x) = 0$ for all $x \in X$, (ii) $d(x, y) = 1$, for all $y \in X, y \neq x$. Prove that every subset of X is open.

(1.3) Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Prove that f is also continuous.

Q2 (25 points). **Functions**

(2.1) Prove the following theorem: "Let $f : S \to T$ be a function from a metric space (S, d_S) into another (T, d_T). Assume that f is one-to-one on S. If S is compact and f is continuous on S, then f^{-1} is continuous (i.e., f is a homeomorphism)."

(2.2) Let $f : [a, b] \to \mathbb{R}$ be a continuous and strictly increasing mapping. Show that its inverse is also continuous and strictly increasing.

(2.3) Prove the following theorem regarding homeomorphisms: "Let $f : X \to Y$ be a function from a metric space (X, d_X) into another (Y, d_Y). Denote by g the inverse of f (that is, $g = f^{-1}$). Then, the following three statements are equivalent:
  (1) f and g are continuous (f is a homeomorphism),
  (2) A set $A \subseteq X$ is open if, and only if, $f(A) \subseteq Y$ is open, and
  (3) A set $B \subseteq X$ is closed if, and only if, $f(B) \subseteq Y$ is closed."

(2.4) Let $f : S \to T$ be a mapping from a metric space (S, d_S) into another (T, d_T). Assume f is one-to-one and is such that

$$d_T(f(x), f(y)) = d_S(x, y), \text{ for all } x, y \in S.$$

Show that f is a homeomorphism (that is, f and f^{-1} are continuous).

(2.5) Given a function $f : \mathbb{R} \to \mathbb{R}$, define two sets A and B in $\mathbb{R}^2$ as follows:

$$A = \{(x, y) \in \mathbb{R}^2 : y < f(x)\}, \text{ and } B = \{(x, y) \in \mathbb{R}^2 : y > f(x)\}.$$

Prove that f is continuous on $\mathbb{R}$ if, and only if, both A and B are open subsets or $\mathbb{R}^2$.

Q3 (15 points). **Differentiation**

(3.1) Consider a function $f : \mathbb{R} \to \mathbb{R}$. Assume f has finite derivative in the half-open interval $0 < x \leq 1$ and such that $|f'(x)| < 1$. Define the sequence $\{f(1/n)\}_{n \in \mathbb{N}}$.
Show that $lim_{n \to \infty} f(1/n)$ exists in $\mathbb{R}$.

(3.2) Let $f : [a, b] \to [a, b]$. Assume f has finite derivative in (a, b), is continuous on $[a, b]$, and for all $x \in (a, b) : |f'(x)| \leq \alpha < 1$, for some $\alpha \in \mathbb{R}_+$.
Show that f has a unique fixed point.

Part II (50 points)

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1. Show that a continuous function defined on a compact set is uniformly continuous.

2. Let $C, X \subset \mathbb{R}^n$. Show that if C is connected, $C \cap X \neq \emptyset$ and $C \cap X^c \neq \emptyset$, then $C \cap \partial X \neq \emptyset$.

3. Let $U \subset \mathbb{R}^m$ be an open set. State, precisely, the definition of "$f : U \to \mathbb{R}^n$ is differentiable at $a \in U$".

4. Let $f : U \to \mathbb{R}^n$ be a Lipschitz-continuous function defined on the open set $U \subset \mathbb{R}^m$, with $a \in U$, and let $g : V \to \mathbb{R}^p$ be a differentiable function defined on the open set $V \subset \mathbb{R}^n$, with $f(U) \subset V$ and $b = f(a)$. Show that if $Dg(b) = 0$ then $g \circ f : U \to \mathbb{R}^p$ is differentiable at a and $D(g \circ f)(a) = 0$.

5. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. Show that if a diffeomorphism[1] $f : U \to V$ exists, then $m = n$. Also, show that if a diffeomorphism $f : U \to V$ is of class C^k ($k \geqslant 1$), then the inverse diffeomorphism f^{-1} is also of class C^k.

6. Let $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping and let $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$ be such that $f(a, b) = b$. For each $x \in \mathbb{R}^m$, consider the function $F^x : \mathbb{R}^n \to \mathbb{R}^n$, defined as $F^x(y) = f(x, y)$, $\forall y \in \mathbb{R}^n$. Suppose that 1 is not an eigenvalue[2] of $DF^a(b) : \mathbb{R}^n \to \mathbb{R}^n$. Prove that there exist open sets U and V, with $a \in U \subset \mathbb{R}^m$ and $b \in V \subset \mathbb{R}^n$, and a C^1 function $h : U \to V$, such that for every $x \in U$, $h(x)$ is the unique fixed point[3] of F^x in V.

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[1]Recall that a diffeomorphism is a differentiable bijection with differentiable inverse.

[2]A real number λ is an *eigenvalue* of a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ if $T \cdot \mathbf{v} = \lambda \mathbf{v}$, for some $\mathbf{v} \neq \mathbf{0}$.

[3]A *fixed point* of a function $g : X \to X$ is a point $p \in X$ such that $g(p) = p$.

Part III (50 points)

Exercise 1. (5 points) Prove the following theorem: "Let $X \in \mathbb{R}^n$ be convex. If $f : X \to \mathbb{R}$ is differentiable and concave, then f is pseudo-concave".

Exercise 2. (10 points) State and prove a Uniqueness of Optimal Solution Theorem.

Exercise 3. Analyze the Household's Utility Maximization problem:

$$\max u(x) \tag{1}$$

$$\text{subject to } w - px \geq 0. \tag{2}$$

1. X is the consumption set;
2. $x = (x_1, x_2, \ldots, x_G) \in X$ is the consumption bundle;
3. $p = (p_1, p_2, \ldots, p_G)$ is the price vector;
4. $u : \mathrm{X} \to \mathbb{R}$ is the utility function of the household;
5. w is the wealth of the household.

Maintained Assumptions:

1. $X = \mathbb{R}^G_+$;
2. $p >> 0$ ($p \in \mathbb{R}^G_{++}$);
3. $w > 0$ ($w \in \mathbb{R}_{++}$);
4. $u(\cdot)$ is continuous, increasing and quasi-concave.

A perfect answer should include a statement of any theorems you need and the minimum possible additional assumptions for establishing the following with respect to the optimal solution:

- (15 points) Characterization (use Kuhn-Tucker Theorem: Necessity and Sufficiency).
- (15 points) Sensitivity Analysis up to the perturbation equations (use Implicit Function Theorem (IFT)).

Exercise 4. (5 points) State the definition of Random Variable.