ECON 897. Final/Waiver Exam. Thursday, August 31, 2006

Name and Program:

You have 180 minutes (three hours). The exam has 180 points and consists of three parts. Each part has a total of 60 points.

Read carefully and think before you write. Good luck!

Part I (60 points)

Q1 (5 points). **Set Theory**

Let $A \subseteq X$ and $B \subseteq Y$. Prove that $(A \times B)^C = (X \times B^C) \cup (A^C \times Y)$.

Q2 (10 points). **Functions**

(2.1) Consider $f : A \to B$. For any $X \subseteq A$ and $Y \subseteq B$, show that
$f^{-1}(Y^C) = (f^{-1}(Y))^C$.
(2.2) Let $A, B \subseteq X$ and $\phi_A : X \to \mathbb{R}$ (characteristic function), where:
$\phi_A(x) = 1$, if $x \in A$,
$\phi_A(x) = 0$, otherwise.
Show that:
(2.2.1) $\phi_A = 1 - \phi_{(A^C)}$,
(2.2.2) $\phi_{(A\cup B)} = max\{\phi_A(x), \phi_B(x)\} = \phi_A(x) + \phi_B(x) - \phi_A(x)\phi_B(x)$, and
(2.2.3) $\phi_{(A\cap B)} = min\{\phi_A(x), \phi_B(x)\} = \phi_A(x)\phi_B(x)$,

Q3 (20 points). **Topology**

(3.1) Let S and T be subsets of $\mathbb{R}$. Define $int(S)$ and show that:
(3.1.1) $int(S) \cap int(T) = int(S \cap T)$, and
(3.1.2) $int(S) \cup int(T) \subseteq int(S \cup T)$.
(3.2) Let X, Y be subsets of $\mathbb{R}$. Define $\overline{X}$ (closure of X) and show that:
(3.2.1) $X \subseteq \overline{X}$,
(3.2.2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, and
(3.2.3) $\overline{\overline{A}} = \overline{A}$.
(3.3) For $A \subseteq \mathbb{R}$, show that $(int(A))^C = \overline{A^C}$

Q4 (25 points). **Continuity and Differentiability**

(4.1) Show that $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$, all $A \subseteq X$.
(4.2) Let X be the set of continuous functions $f : [a, b] \to \mathbb{R}$. Let d^* be the distance function on X defined by $d^*(f, g) = \int_a^b |f(t) - g(t)|dt$, for $f, g \in X$.
For each $f \in X$, set $I(f) = \int_a^b f(t)dt$. Prove that I is a continuous function.
(4.3) Let $f : [a, b] \to [c, d]$ be a continuous and increasing function.
(4.3.1) Prove that f is a homeomorphism (f and f^{-1} are continuous functions).
(4.3.2) Give sufficient conditions to ensure that f is a diffeomorphism (f and f^{-1} are C^1 functions) – the weaker your conditions the better.

Part II (60 points)

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1. Let X be a nonempty metric space with metric $d : X \times X \to R$ and $a \in X$.

(1) Show that if K is a compact subset of X, then K is closed.

(2) From (1), we know that if $X = R^n$, then $B_1(a) = \{x \in X : d(a,x) < 1\}$ is not compact, because it is not closed. Is it true for any metric space X that $B_1(a) = \{x \in X : d(a,x) < 1\}$ is not compact? If so, prove it. If not, provide a counterexample.

(3) Suppose $X = R^n$. Then $\overline{B}_1(a) = \{x \in X : d(a,x) \leq 1\}$ is compact in X with a metric $d_k(x,y) = \left(\sum_{i=1}^n (x_i - y_i)^k\right)^{1/k}$ for any $k \in \{1, 2, ...\} \cup \{\infty\}$. Is it true that $\overline{B}_1(a)$ is compact, independent of the metric? That is, is $\overline{B}_1(a)$ compact no matter what metric you use? If so, prove it. Otherwise, provide a counterexample and explain why $\overline{B}_1(a)$ is not compact.

2. X, Y : metric spaces, X : compact and $f : X \to Y$, C^0.

(1) Prove that $f(X)$ is compact.

(2) (Extreme Value Theorem) Suppose $Y = R$. Prove that if $f(X)$ is compact, then $\exists x \in X$ such that $f(x) \geq f(z), \forall z \in X$.

(3) (The Homeomorphism Theorem) Prove that if f is bijective, then f^{-1} is also continuous. (Hint : use (1) and a topological definition of continuity)

3. Consider a function $f : R^3 \to R^2$ such that

$$f(x_1, x_2, x_3) = \begin{pmatrix} x_1 x_2 + x_3 \\ (x_1 + x_2)^{x_3} \end{pmatrix} \tag{1}$$

Find the derivative of f at $a = (a_1, a_2, a_3)$.

4. Consider the function $f : R^2 \to R$ defined by $f(x,y) = x^2 + y^2 - 1$. Let $M = \{(a,b) \in R^2 : f(a,b) = 0\}$ and $A = \{(a,b) \in R^2 :$ for some open sets U, V (in R) which contains a and b respectively, there exists a function $g : U \to V$ s.t. $f(x, g(x)) = 0, \forall x \in U\}$. Show that A is generic in M[1].

5. Let $f : [a,b] \to R$ be an increasing function. Show that $\{x : f$ is discontinuous at $x\}$ has measure 0.

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[1] A is generic in X if A is open in X and $X \backslash A$ has zero measure.

Part III (60 points)

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1. **(Concavity)** Let $X \subset \mathbb{R}^k$. Prove that $f : X \to \mathbb{R}$ is quasi-concave if and only if $U_\alpha = \{x \in X : f(x) \geq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

2. **(Regularity)** Let $X \subset \mathbb{R}^k$ be compact. Suppose that $f : X \times A \to \mathbb{R}$ is continuous. Suppose that $\Gamma : A \rightrightarrows X$ is such that $\Gamma(\alpha) = X$ for all $\alpha \in A$.

   (a) Prove that Γ is upper hemi-continuous.

   (b) Prove that Γ is lower hemi-continuous.

   (c) Define $x^* : A \rightrightarrows X$ such that $x^*(\alpha) = \arg\max_{x \in \Gamma(\alpha)} f(x, \alpha)$.Prove that x^* is well-defined and upper hemi-continuous.

3. **(Characterization)**

   (a) Let $X \subset \mathbb{R}^n$ be open and convex. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}^m$.Consider the canonical programming problem.

$$\begin{array}{cc} \max & f(x) \\ subj.to & g(x) \geq 0 \;\; (P) \\ & x \in X \end{array}$$

   State the Kuhn-Tucker Necessity and Sufficiency Theorems.

   (b) Let the household's consumption set $X \subset \mathbb{R}^G_+$ be open and convex. Let the household's utility function $u : X \to \mathbb{R}$ be given. Let the household's endowment of resources $e \in X$ be strictly postive. Let the price vector $p \in \mathbb{R}^G_{++}$ be given. Consider the household's problem.

$$\begin{array}{cc} \max & u(x) \\ subj.to & p(e - x) \geq 0 \;\; (H) \\ & x \in X \end{array}$$

   State the minimum set of assumptions guaranteeing that the set of optimal solutions is fully characterized by the Kuhn-Tucker conditions.

   (c) Maintaining these assumptions, show that the set of optimal solutions is fully characterized by the Kuhn-Tucker conditions.

   (d) Use the Kuhn-Tucker conditions to characterize the set of optimal solutions.

4. **(Univariate Normal Distribution)** Let $X \sim \mathcal{N}(0, 1)$.

   (a) Prove that $E(X) = 0$ and that $Var(X) = 1$.

(b) Let $Y = \mu + \sigma X$ for $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_{++}$. Show that $Y \sim \mathcal{N}(\mu, \sigma^2)$.

5. **(Multivariate Normal)** Suppose that

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \right).$$

Show that X and Y are stochastically independent.