ECON 898 - MATHEMATICS FOR ECONOMICS

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PRACTICAL MATTERS

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3. Course's webpage: http://www.ssc.upenn.edu/~davilaj/econ898.html
4. Meetings: Monday and Wednesday, from 10:30 to 12:00 at 395 McNeil.
5. 1st exam: Monday, 0ctober 22, 2001, 10:30-12:00.
6. 2nd exam: Monday, December 10, 2000, 10:30-12:00.
7. Evaluation: 1st exam 45%, 2nd exam 45%, homeworks 10%.

REFERENCES

Handouts with the contents of the course will be posted at the course's website http://www.ssc.upenn.edu/∼davilaj/econ898.html. They are permanently under (hopefully convergent) construction, and are intended to cover all what you will be required to know for the homeworks and the exams. Nevertheless although they are, to a reasonable extent, self-contained and almost every term used is defined within them and most of the statements are proved, it may be extremely helpful to review the contents of the course in different sources in order to get a good understanding of the material that will be covered in it. Please find below a list of references useful for such a review.

The handouts will consist mostly of a mix of definitions, theorems and examples, with expanded or more neat proofs when necessary for a better understanding, and a unified notation and terminology, all arranged in an ordered sequence that (almost) avoids using any term undefined at any stage and builds (almost) every theorem on previous theorems.

Typeset by $\mathcal{AMS}$-TEX

Basic references in real analysis (quite useful),

1. Michel and Herguet, *Applied Algebra and Functional Analysis*, Dover, ISBN 0-486-67598.
2. Haaser and Sullivan, *Real Analysis*, Ch. 1, 2, 3, 4, 5, 7, 11 Dover, ISBN 0-486-66509-7.
3. Kolmogorov and Fomin, *Introductory Real Analysis*, Ch. 1, 2, 4, 5, Dover, ISBN 0-486-61226-0.

References oriented to graduate students in Economics:

1. Moore, James C., *Mathematical Methods for Economic Theory*, Springer, ISBN 3-540-66235-9.
2. Efe A. Ok, *Real Analysis and Probability with Economic Applications*, Ch. 2, 3, 4, 5, unpublished lecture notes, May 2000.
3. Takayama, *Mathematical Economics*, Ch. 5, 8 Cambridge University Press, ISBN 0-52131-498-4.
4. Stokey and Lucas, *Recursive Methods in Economic Analysis*, Ch. 3, 4, 17, Harvard University Press, ISBN 0-674-75096-9.

Advanced references in real analysis (useful but with extensive parts well beyond the scope of the course):

1. Aliprantis and Border, *Infinite Dimensional Analysis*, Ch. 1, 3, 6, 15, 16, Springer, ISBN 3-540-65854-8.
2. Aliprantis and Burkinshaw, *Principles of Real Analysis*, Ch.1 Academic Press, ISBN 0-12-050257-7.

Overview of the course

This course is intended to acquaint you with some of the basic higher mathematics that you need to proceed with the courses and research leading you to obtaining the Ph.D. degree in Economics. Basically, we will review the mathematics needed to get to two of the theorems driving basic results in economics, i.e. Bellman's principle of optimality and Kakutani's fixed point theorem. In effect, these results will allow you to solve dynamic programing problems arising for instance in macroeconomics, as well as to obtain the existence of equilibria both in games and economies. Finally, basic theorems of differential calculus will allow for results on at least local uniqueness of equilibria and comparative statics whenever the framework allows it (which is the case in the economies and also in some games).

Thus, after an introduction intended to motivate the need for more advanced mathematics in order to deal with simple economic problems, section 2 introduces the basic concept of relation and from it we introduce the concepts of function as a special type of relation, and of correspondence as a set-valued function. From here, sections 3, 4, 5, and 6 build on the introduction of a notion of distance between the elements of a set, leading to the concept of metric space. Section 4 presents namely the notions of continuity of functions from a metric space to another metric space, and of contraction. Contractions have a very important property when they operate on metric spaces that have the property of being "complete", a property the spaces we deal with fortunately have: in complete metric spaces contractions have a point of the space mapped into itself and this point is unique. The importance of such result, known as the Contraction Mapping Theorem or Banach's Fixed Point Theorem, is that it allows to prove the existence of solutions to a functional equation like Bellman's (under the adequate assumptions), which is the bread and butter of current macroeconomics and many other fields in economics. Not only this result guarantees the existence of such solution, but it provides also a way to compute numerical approximations of the solution and other related functions.

Section 5 deals with correspondences and some notions of continuity available for them, paving the way to the statement of a property, the Maximum Theorem, which establishes good properties for the dependence of the set of maximizers of a real-valued function and the maximum of the function on the parameters determining its domain. This result is then used to show Bellman's principle of optimality that allows to reduce the standard dynamic optimization problem to Bellman's equation, whose solution we already know how to find (at least approximately and under the adequate conditions).

From here the course takes different direction. We need to introduce more structure on top of endowing the space with a metric in order to get more results. This structure is that of a vector space, which allow us to "operate" with the elements of a set (conveniently renamed *vectors* for that purpose) very much in the same way we operate with numbers. Based on these operations, the notions of linear combination of vectors, convexity, and linear dependence and independence of a set are introduced, as well as the idea of basis, i.e. of a "minimal" subset of points of the space with which we are able to generate the entire space by means of these operations, in such a way that having a basis amounts to having the space. Many of the properties of a vector space depend on whether it has a finite basis or not. The number of elements in a basis (every basis happens to have the same number of them) is the dimension of the vector space, and finite- versus infinite-dimensional vector spaces exhibit different properties.

The interest of the vector space structure comes, on the one hand, from the notion of convexity, i.e. the property some sets have of containing any convex linear combination of their elements. Once we have the notion of convexity, we will be in a position to get another application of the Maximum's Theorem in the context of finite-dimensional real vector spaces: a version of the Brower's Fixed Point Theorem based on a particular version of the same theorem specialized to the unit ball (later we may have hopefully have time to provide a proof of this last result as well, thus closing the argument). We will finish this part getting Kakutani's Fixed Point Theorem from Brower's, and then from Kakutani's we will obtain Nash's theorem of the existence of an equilibrium for games of finitely many players with continuous payoffs (quasiconcave in their own strategies) and compact, convex, finite-dimensional strategy sets. Kakutani's theorem is also at the heart of the existence of a Walrasian equilibrium of an economy of finitely many consumers and producers with convex preferences and technologies.

On the other hand, the interest of the vector space structure comes also from the existence of a type of functions between vector spaces, the linear functions introduced in Section 8, extremely useful to approximate locally most other functions (the useful ones, by the very fact that they can be approximated locally). Focusing on the idea of approximation, notice that it calls for the introduction of a metric in each vector space. The way to introduce a metric consistently with the already existing vector space structure is through the notion of norm, which gives a "length" to each of the elements of the vector space. In effect, each norm induces a metric by means of setting the length of the difference between any two vector as the distance between them (a richer way to metrize a vector space is to endow it with an inner product, a way to operate vectors that not only induces a norm and hence a metric, but also allows to define "angles" between vectors, carrying with it all the geometric intuitions of the usual space into abstract vector spaces, an extremely useful fact to "visualize" results). Thus section 11 deals with the issue of continuity of linear functions, a matter that relies heavily on the dimension of the vector space on which they are defined.

The full benefits of the structure of normed vector spaces are obtained exploiting the idea of approximating functions locally by means of linear functions. The functions for which such approximation is possible are known as differentiable functions. These functions and their properties are presented in Section 12, among them the familiar results of multivariate differential calculus as well as the extremely useful Implicit Function Theorem in the most general set up. The Implicit Function Theorem guarantees allows to see the inverse image of a (regular) value of a function (or the set of solutions to a system of equations, in the finite-dimensional case) as a function itself, which is at the foundations of any comparative statics-like argument. In order to establish this theorem we will make use again of the existence and uniqueness of a fixed point of a contraction. The Inverse Function Theorem is presented as a corollary of the Implicit Function Theorem. Finally, the fact that the Implicit Function Theorem allows to see as graphs of differentiable functions some subsets of vector spaces makes sense of the idea of a tangent space to a "surface" at a point in an abstract setup. This property is exploited to characterize the maxima and minima of real-valued functions defined on constrained domains (the familiar Lagrange and Kuhn-Tucker characterizations will arise as an easy consequence of these characterizations).

List of topics

(1) **Introduction**

(2) **Relations, orders and functions**

Relation from a set to a set. Inverse of a relation. Function from a set to a set. Set of functions from a set to a set. Image and inverse image a set by a relation. Range and domain of a relation. Injective, surjective and bijective relations. Composition of relations. Extension and restriction of a relation. Binary relations. Upper and lower bound, greatest and least element, supremum and infimum, maximum and minimum, maximal and minimal of a set by a binary relation. Reflexivity, antireflexivity, symmetry, asymmetry, antisymmetry, transitivity, negative transitivity, completeness, totality of a binary relation. Equivalence relations. Preorders. Partial orders Complete orders. *The complete partial order of real numbers.* Extension of a function. Increasing functions between completely ordered sets. Power set of a set. Correspondence from a set to a set. Graph of a correspondence. Sequence in a set. Subsequence of a sequence. Finite sets.

(3) **Metric spaces**

Metrics. Metric spaces. *Discrete spaces.* l_p^n, ℓ_∞^n, ℓ_p *and* ℓ_∞ *spaces. The space of bounded real valued functions with the sup metric.* Subspaces. Closure, limit and interior points. Closure and interior of a set. Open sets and closed sets. Bounded sets. Connected,separable, compact, complete metric spaces. Compactness and sequential compactness. Sequences in metric spaces. Convergence. Limit of a convergent sequence. Cauchy sequences. *Sequences in* $\mathbb{R}$*, in* $\mathbb{R}^n$*, in* ℓ_p *and* ℓ_∞*.* Totally bounded sets.

(4) **Functions between metric spaces**

Limit of a function at a closure point of its domain. Uniqueness of the limit. Characterization of the limit by sequences. Continuity. Continuity and separability, completeness and connectedness. Homeomorphisms. Uniform continuity. Contractions. Existence and uniqueness of fixed points of contractions on complete metric spaces. Sufficient characterizations of contractions. Blackwell's sufficient condition. Solutions to functional equations. *Bellman's functional equation.* Lipschitz functions. Continuous functions and compact sets.

(5) **Correspondences between metric spaces**

Upper, lower hemicontinuity and continuity of a correspondence between metric spaces. Complete characterization of lower hemicontinuous correspondences by sequences. Complete characterization of upper hemicontinuous compact valued correspondences by sequences. Upper hemicontinuity of every correspondence taking values in a compact metric space and with closed graph. Closed graph of every closed-valued upper hemicontinuous correspondence. Stability of upper hemicontinuity under unions. Stability of upper hemicontinuity and compact-valuedness under intersections. Stability of local upper hemicontinuity and compact-valuedness under intersections with closed-valued correspondences. Compactness of the image of a compact by an upper hemicontinuous compact valued correspondence. Complete characterization of lower hemicontinuous correspondences by sequences.

(6) **The Maximum Theorem and its applications**

The Maximum Theorem. Bellman's principle of optimality for the standard dynamic optimization problem. (A version of) Brower's fixed point theorem.

(7) **Vector Spaces**

Sums and multiplications of scalars. *The field of real numbers. The field of complex numbers.* Vector spaces. *The real vector space of* n*-tuples of real numbers. The real vector spaces of polynomials with real coefficients. The vector space of* $m \times n$ *matrices of scalars.* Vector subspaces. Linear combinations of vectors. Convex linear combinations. Convex sets. Linearly dependent and independent sets of vectors. Bases. Unique decomposition with respect to a basis. Finite dimensional vector spaces. Dimension. The vector space of n-tuples of scalars. *The finite dimensional real vector space* $\mathbb{R}^n$*.* Infinite dimensional vector spaces. *The infinite dimensional vector spaces of sequences of real numbers, of all the real polynomials, of real valued functions over a non finite domain.*

(8) **From Brower's to Kakutani's fixed point theorem and Nash's theorem**

Approximation of correspondences. Von Neumann approximation lemma. Michael's selection theorem. Brower's fixed point theorem for the closed unit ball of a normed real vector space. Brower's fixed point theorem for compact, convex set of a normed real vector space. Kakutani's fixed point theorem. Nash equilibrium existence theorem for games of finitely many players with compact, convex, finite-dimensional strategy sets and continuous payoffs quasi concave in their own strategies.

(9) **Linear functions between real vector spaces**

Linear functions. The vector subspaces formed by the range and the null space of a linear function. The complete characterization of linear functions by their values at any basis. Linearity of the composition of linear functions. Complete characterization of injective linear functions. Inverse of an injective linear function. Isomorphisms. Inverse of an isomorphism. Equivalence by isomorphism of all the n-dimensional vector spaces over a given field. Dimension induced by an isomorphism. Linearity of sums and multiplication by a scalar of linear functions Vector spaces of linear functions. The identification of vector spaces of matrices of scalars and of linear functions. Matrix of the composition of linear functions between finite dimensional vector spaces Matrix of a linear function between finite dimensional spaces after a change of bases.

(10) **Normed real vector spaces**

Metric induced by a norm. Norm of the Cartesian product of two normed real vector spaces.

(11) **Inner product spaces**

Cauchy-Schwarz inequality in inner product spaces. Norm induced by an inner product.

(12) **Linear functions between normed real vector spaces**

Characterization of continuous linear functions by their continuity at the null vector. Characterization of the convergence of a sequence in a finite dimensional normed real vector space by the convergence of coordinate sequences. Continuity of linear functions with finite dimensional domains. Sup norm in finite dimensional real vector spaces. Equivalence of continuity and bounded-linearity of linear functions. Stability of boundedness by continuous linear transformations. The vector subspaces of bounded-linear or continuous functions. Norm of a continuous linear function. Equivalent norms. Equivalence relation between equivalent norms. Equivalence of all the norms in a finite dimensional space. Equivalence of convergence with respect to equivalent norms. Equivalence by homeomorphism of every n-dimensional real vector space.

(13) **Differentiable functions between normed real vector spaces**

Differential of a function at a point. Uniqueness of the differential. Equivalence between continuity and continuity of the differential of differentiable functions. Rules of differentiation. *Differential of a constant function, of a linear function, of the sum of functions, of the multiplication by a scalar of function, of the composition of functions.* Properties of differentiable real valued functions on the real line. *Rolle's theorem. Generalized mean value theorem. Mean value theorem.* Directional derivatives. Directional derivatives as values of differentials. Partial derivatives. Differentials in terms of partial derivatives. Differentiability at any point of an open implied by the continuity of partial derivatives in the open. Equivalence of continuous differentiability and continuity of the partial derivatives in an open. Jacobian matrix, gradient vector. Mean value theorem for functions on the real line taking values in arbitrary normed real vector spaces. Mean value theorem for functions between arbitrary normed real vector spaces. Partial functions. Differentiability of partial functions induced by the differentiability of a function on a cartesian product. Partial differentials. Continuity of partial differentials induced by the continuity of the differential. Implicit function theorem in arbitrary normed real vector spaces. Inverse function theorem in arbitrary normed real vector spaces. Tangent space to a function at a point where it is differentiable.