ESTIMATING THE QUADRATIC COVARIATION MATRIX FROM NOISY OBSERVATIONS: LOCAL METHOD OF MOMENTS AND EFFICIENCY

By Markus Bibinger[*], Nikolaus Hautsch[*], Peter Malec[*] and Markus Reiss[*]

Humboldt-Universität zu Berlin

An efficient estimator is constructed for the quadratic covariation or integrated covolatility matrix of a multivariate continuous martingale based on noisy and non-synchronous observations under high-frequency asymptotics. Our approach relies on an asymptotically equivalent continuous-time observation model where a local generalised method of moments in the spectral domain turns out to be optimal. Asymptotic semiparametric efficiency is established in the Cramér-Rao sense. Main findings are that non-synchronicity of observation times has no impact on the asymptotics and that major efficiency gains are possible under correlation. Simulations illustrate the finite-sample behaviour.

1. Introduction. The estimation of the quadratic variation or integrated volatility of a semi-martingale is a key question, both from a theoretical viewpoint as well as for applications, particularly in finance. Here we treat the multi-dimensional case, where the quadratic covariation or integrated covolatility matrix is the quantity of interest. It turns out that the richer geometry, e.g. due to non-commuting matrices, generates new effects and calls for a deeper mathematical understanding. Covariation estimates are particularly important in various financial applications, for instance, as inputs in portfolio allocation problems, quantification of risk, hedging or asset pricing. The availability of high-frequency data opens up new ways for inference. As the data is typically polluted by observational noise, e.g. by microstructure frictions, estimation in these models is far from obvious and furnishes unexpected results.

We focus on the fundamental statistical model where the d-dimensional discrete-time process

$$(\mathcal{E}_0) \qquad Y_i^{(l)} = X_{t_i^{(l)}}^{(l)} + \varepsilon_i^{(l)}, 0 \le i \le n_l, 1 \le l \le d,$$

is observed with the d-dimensional continuous martingale

$$X_t = X_0 + \int_0^t \Sigma^{1/2}(s)\, dB_s,\ t \in [0,1],$$

in terms of a d-dimensional standard Brownian motion B and the squared (instantaneous or spot) covolatility matrix

$$\Sigma(t) = (\Sigma_{lr}(t))_{1 \le l, r \le d} \in \mathbb{R}^{d \times d}.$$

The signal part X is assumed to be independent of the observation noise ε. The observation errors $(\varepsilon_i^{(l)}), 1 \le l \le d, 1 \le i \le n_l$, are assumed to be mutually independent and centered normal with variances η_l^2. The observation times are given via quantile transformations as $t_i^l = F_l^{-1}(i/n_l)$ for some distribution functions F_l. While this model is certainly an idealisation of many real data situations, its precise analysis delivers a profound understanding and thus serves as a basis for developing procedures in more complex models.

Covariation estimation is a core research topic in current financial econometrics and various approaches exist. Let us mention the quasi-maximum-likelihood method by Aït-Sahalia $et\ al.$ [1], realised kernels by Barndorff-Nielsen $et\ al.$ [3], pre-averaging by Christensen $et\ al.$ [5] and the local spectral estimator by Bibinger and Reiß [4]. In contrast to the univariate case, however, the asymptotic properties are very involved, difficult to compare and a lower efficiency bound was lacking as a benchmark.

Building on the idea of locally constant approximations, we propose a local method of moments (LMM) estimator in the spectral domain which is shown to be asymptotically efficient. We perform an asymptotic analysis where the sample sizes $n_1, \ldots, n_d$ tend to infinity. In Section 2 (strong) asymptotic equivalence in Le Cam's sense is established by Theorem 2.4 with the signal-in-white-noise model

$$(\mathcal{E}_1) \qquad dY_t = X_t\, dt + \mathrm{diag}(H_{n,l}(t))_{1 \le l \le d}\, dW_t, t \in [0,1],$$

where W is a standard d-dimensional Brownian motion independent of B and the local noise level is given by

$$H_{n,l}(t) := \eta_l (n_l F_l'(t))^{-1/2}. \qquad (1.1)$$

The imposed regularity condition is that $\Sigma(t)$ is the sum of an L^2-Sobolev function of regularity β and an L^2-martingale and the size of β accommodates for asymptotically separating sample sizes $(n_l)_{1 \le l \le d}$.

Let us recall that if two sequences of statistical experiments are asymptotically equivalent, then any statistical procedure in one experiment has a counterpart in the other experiment with the same asymptotic properties for bounded loss functions, see Le Cam and Yang [17] for a thorough introduction. Our proof is constructive such that the procedure that we shall develop for $(\mathcal{E}_1)$ has a concrete counterpart in $(\mathcal{E}_0)$ with the same asymptotic properties.

A remarkable theoretical consequence of this result is that under noise the asynchronicity of the data does not affect the asymptotically efficient procedures (it is of smaller asymptotic order). In model $(\mathcal{E}_1)$ the distribution functions F_l only generate a varying local noise level $H_{n,l}(t)$, but the shift between observation times of different processes does not matter. This is in sharp contrast to the noiseless setting where the variance of the Hayashi-Yoshida estimator [12] suffers from errors due to asynchronicity, which carries over to the preaveraged version by Christensen *et al.* [5] designed for the noisy case.

In Section 3 we consider the continuous-time model $(\mathcal{E}_1)$ and go over to a block-wise constant approximation. Empirical Fourier coefficients yield local spectral statistics (S_{jk}) in the spirit of Reiß [19]. On each block we apply locally a generalised method of moments, using a weighted sum of the empirical covariance matrices $S_{jk} S_{jk}^\top \in \mathbb{R}^{d \times d}$ and a bias correction. The optimal weighting for estimating an entry of the covariation matrix combines in general all entries of $S_{jk} S_{jk}^\top$. The non-commutativity of different Fisher information matrices then implies in particular that the volatility estimation for one coordinate process $X^{(l)}$ gains in efficiency when using data for all other potentially correlated processes $X^{(r)}$, see Sections 4.2 and 5 for details and the improvement with respect to the approach in Bibinger and Reiß [4]. Note the contrast with i.i.d. observations of a Gaussian vector where the empirical variance of one component is an efficient estimator and using the other entries cannot improve the variance estimator unless the correlation is known, cf. the classical Example 6.6.4 in Lehmann and Casella [18].

In Theorem 3.2 a multivariate central limit theorem (CLT) is provided for an oracle LMM (Local Method of Moments) estimator, using the unknown optimal weights and an information-type matrix for normalisation. Specifying to sample sizes of the same order n, Corollary (3.3) yields a CLT with rate $n^{1/4}$ and a covariance structure between matrix entries, which is given explicitly by concise matrix algebra. Using pre-estimated weight matrices, a

fully adaptive version of the LMM-estimator is obtained, which by Theorem 3.4 shares the same asymptotic properties as the oracle estimator.

Another main result of this work is that the asymptotic covariance structure of the LMM-estimators is optimal in a semiparametric Cramér-Rao sense. In Section 4 a lower bound proof is achieved by a combination of space-time transformations and advanced calculus for covariance operators. The concrete form of the asymptotic variance is discussed for some key settings, thus generalising the univariate optimality theory by Gloter and Jacod [10] and Reiß [19] to the multivariate case. The discretisation and implementation of the estimator for model $(\mathcal{E}_0)$ is briefly described in Section 5 and presented together with some numerical results for a simple toy model and a more complex and realistic scenario. The finite sample behaviour of the LMM estimators is well predicted by the asymptotic theory (even in cases where it does not apply formally) and some comparison with competing procedures is provided.

2. From discrete to continuous-time observations.

2.1. *Setting.* First, let us specify different regularity assumptions. For functions $f : [0,1] \to \mathbb{R}^m$, $m \geq 1$ or also $m = d \times d$ for matrix values, we introduce the L^2-Sobolev ball of order $\alpha \in (0,1]$ and radius $R > 0$

$$H^\alpha(R) = \{f \in H^\alpha([0,1],\mathbb{R}^m) | \|f\|_{H^\alpha} \leq R\} \text{ where } \|f\|_{H^\alpha} := \max_{1 \leq i \leq m} \|f_i\|_{H^\alpha},$$

which for matrices means $\|f\|_{H^\alpha} := \max_{1 \leq i,j \leq d} \|f_{ij}\|_{H^\alpha}$. We also consider Hölder spaces $C^\alpha([0,1])$ and Besov spaces $B^\alpha_{p,q}([0,1])$ of such functions. Canonically, for matrices we use the spectral norm $\| \cdot \|$ and we set $\|f\|_\infty := \sup_{t \in [0,1]} \|f(t)\|$.

In order to pursue asymptotic theory, we impose that the deterministic samplings in each component can be transferred to an equidistant scheme by respective quantile transformations independent of $n_l, 1 \leq l \leq d$.

ASSUMPTION 2.1.-(α) *Suppose that there exist differentiable distribution functions $F_l \in C^\alpha([0,1])$, $1 \leq l \leq d$, with $F_l(0) = 0$, $F_l(1) = 1$ and $F_l' > 0$, such that the observation times in $(\mathcal{E}_0)$ are generated by $t_i^{(l)} = F_l^{-1}(i/n_l)$, $0 \leq i \leq n_l$, $1 \leq l \leq d$.*

We gather all assertions on the instantaneous covolatility matrix function $\Sigma(t)$, $t \in [0,1]$, which we shall require at some point.

ASSUMPTION 2.2.   *Let* $\Sigma : [0,1] \to \mathbb{R}^{d \times d}$ *be a possibly random function with values in the class of symmetric, positive semi-definite matrices, independent of* X *and the observational noise, satisfying:*

(i-β) $\Sigma \in H^{\beta}([0,1])$ *for* $\beta > 0$.
(ii-α) $\Sigma = \Sigma^B + \Sigma^M$ *with* $\Sigma^B \in B_{1,\infty}^{\alpha}([0,1])$ *for* $\alpha > 0$ *and* Σ^M *a matrix-valued* L^2*-martingale.*
(iii-$\underline{\Sigma}$) $\Sigma(t) \geq \underline{\Sigma}$ *for a strictly positive definite matrix* $\underline{\Sigma}$ *and all* $t \in [0,1]$.

Let us briefly discuss the different function spaces, see e.g. Cohen [7, Section 3.2] for a survey. First, any α-Hölder-continuous function lies in the L^2-Sobolev space H^{α} and any H^{α}-function lies in the Besov space $B_{1,\infty}^{\alpha}$, where differentiability is measured in an L^1-sense. The important class of bounded variation functions (e.g., modeling jumps in the volatility) lies in $B_{1,\infty}^1$, but only in H^{α} for $\alpha < 1/2$. In particular, part (ii-α), $\alpha \leq 1$, covers L^2-semi-martingales by separate bounds on the drift (bounded variation) and martingale part. Beyond classical theory in this area is the fact that also non-semi-martingales like fractional Brownian motion B^H with hurst parameter $H > 1/2$ give rise to feasible volatility functions in the results below, using $B^H \in C^{H-\varepsilon} \cap B_{1,\infty}^H$ for any $\varepsilon > 0$ from Ciesielski *et al.* [6].

In the sequel the potential randomness of Σ is often not discussed additionally because by independence we can always work conditionally on Σ. Finally, let us mention that we could also weaken the Hölder-assumptions on $F_1, \ldots, F_d$ towards Sobolev or Besov regularity at the cost of tightening the assumptions on Σ. For the sake of clarity this is not pursued here.

Throughout the article we write $Z_n = \mathcal{O}_P(\delta_n)$ and $Z_n = \mathcal{o}_P(\delta_n)$ for a sequence of random variables Z_n and a sequence δ_n, to express that $\delta_n^{-1} Z_n$ is bounded or tends to zero in probability, respectively. Analogously $\mathcal{O}$ (or equivalently $\lesssim$) and $\mathcal{o}$ refer to deterministic sequences. We write $Z_n \asymp X_n$ if $Z_n = \mathcal{O}_P(Y_n)$ and $Y_n = \mathcal{O}_P(Z_n)$ and the same for deterministic quantities. E_d denotes the d-dimensional unit matrix and $\delta_{p,q} = \mathbb{1}(p = q)$ equals 1 for $p = q$ and 0 otherwise.

2.2. *Continuous-time experiment.*

DEFINITION 2.3.   *Let* $\mathcal{E}_0((n_l)_{1 \leq l \leq d}, \beta, R)$ *with* $n_l \in \mathbb{N}, \beta \in (0,1], R > 0$, *be the statistical experiment generated by observations from* $(\mathcal{E}_0)$ *with* $\Sigma \in H^{\beta}(R)$. *Analogously, let* $\mathcal{E}_1((n_l)_{1 \leq l \leq d}, \beta, R)$ *be the statistical experiment generated by observing* $(\mathcal{E}_1)$ *with the same parameter class.*

As we shall establish next, experiments $\mathcal{E}_0$ and $\mathcal{E}_1$ will be asymptotically

equivalent as $n_l \to \infty, 1 \le l \le d$, at a comparable speed, denoting

$$n_{min} = \min_{1 \le l \le d} n_l \text{ and } n_{max} = \max_{1 \le l \le d} n_l.$$

THEOREM 2.4. *Grant Assumption 2.1-(β) on the design. The statistical experiments $\mathcal{E}_0((n_l)_{1 \le l \le d}, \beta, R)$ and $\mathcal{E}_1((n_l)_{1 \le l \le d}, \beta, R)$ are asymptotically equivalent for any $\beta \in (0, 1/2]$ and $R > 0$, provided*

$$n_{min} \to \infty, \quad n_{max} = \mathcal{O}((n_{min})^{1+\beta}).$$

More precisely, the Le Cam distance Δ is of order

$$\Delta\left(\mathcal{E}_0((n_l)_{1 \le l \le d}, \beta, R), \mathcal{E}_1((n_l)_{1 \le l \le d}, \beta, R)\right) = \mathcal{O}\left(R^2 \left(\sum_{l=1}^{d} n_l/\eta_l^2\right) n_{min}^{-1-\beta}\right).$$

By inclusion, the result also applies for $\beta > 1/2$ when in the remaining expressions β is replaced by $\min(\beta, 1/2)$. A standard Sobolev smoothness of Σ is β almost $1/2$ for diffusions with finitely many or absolutely summable jumps. In that case, the asymptotic equivalence result holds if n_{max} grows more slowly than $n_{min}^{3/2}$. Theorem 2.4 is proved in the appendix in a constructive way by warped linear interpolation, which yields a readily implementable procedure, cf. Section 5 below.

3. Localisation and method of moments.

3.1. *Construction.* We partition the interval $[0, 1]$ in blocks $[kh, (k+1)h)$ of length h. On each block a parametric MLE for a constant model could be sought for. Its numerical determination, however, is difficult and unstable due to the non-concavity of the ML objective function and its analysis is quite involved. Yet, the likelihood equation leads to spectral statistics whose empirical covariances estimate the quadratic covariation matrix. We therefore prefer a localised method of moments (LMM) for these spectral statistics where for an adaptive version the theoretically optimal weights are determined in a pre-estimation step, in analogy with the classical (multi-step) GMM (generalised method of moments) approach by Hansen [11].

As motivated in Reiß [19], consider the $L^2([0,1])$-orthonormal system (φ_{jk}) and its antiderivatives (Φ_{jk}):

$$\varphi_{jk}(t) = \sqrt{\frac{2}{h}} \cos\left(j\pi h^{-1}(t - kh)\right) \mathbb{1}_{[kh,(k+1)h]}(t), j \ge 1, \quad (3.1a)$$

$$\Phi_{jk}(t) = \frac{\sqrt{2h}}{j\pi} \sin\left(j\pi h^{-1}(t - kh)\right) \mathbb{1}_{[kh,(k+1)h]}(t), j \ge 1. \quad (3.1b)$$

In each component local spectral statistics are defined by

$$S^{(l)}_{jk} = \pi j h^{-1} \int_{kh}^{(k+1)h} \varphi_{jk}(t)\, dY^{(l)}(t),\, j \geq 1, k = 0, \ldots, h^{-1}-1, 1 \leq l \leq d, \quad (3.2)$$

from the continuous-time experiment $\mathcal{E}_1$. In order to design our estimator, we consider a locally constant approximation of the general non-parametric model.

DEFINITION 3.1. Set $\bar{f}_h(t) := h^{-1} \int_{kh}^{(k+1)h} f(s)ds$ for $t \in [kh, (k+1)h)$, $k \in \mathbb{N}_0$, a function f on $[0,1]$ and $h \in (0,1)$. Assume $h^{-1} \in \mathbb{N}$ and let $X^h_t = X_0 + \int_0^t \bar{\Sigma}^{1/2}_h(s)\, dB_s$ with a d-dimensional standard Brownian motion B. Define the process

$$(\mathcal{E}_2) \qquad d\tilde{Y}_t = X^h_t\, dt + \operatorname{diag}\left(\sqrt{\overline{H^2}_{n,l,h}(t)}\right)_{1 \leq l \leq d} dW_t\,, t \in [0,1]\,,$$

where W is a standard Brownian motion independent of B and with noise level (1.1). The statistical model generated by the observations from $(\mathcal{E}_2)$ for $\Sigma \in H^\beta(R)$ is denoted by $\mathcal{E}_2((n_l)_{1 \leq l \leq d}, h, \beta, R)$.

In experiment $\mathcal{E}_2$ we thus observe a process with a covolatility matrix which is constant on each block $[kh, (k+1)h)$, $k = 0, 1, \ldots, h^{-1}-1$, and corrupted by noise of block-wise constant magnitude. Our approach is founded on the idea that for small block sizes h and sufficient regularity this piecewise constant approximation is close to $\mathcal{E}_1$.

A basic ingredient to derive the covariance structure are the joint moments for a d-dimensional random vector $X \sim \mathbf{N}(0, Q)$. For $(l, r, p, q) \in \{1, \ldots, d\}^4$ we have by Isserlis [13]:

$$\mathbb{E}\left[X^{(l)}X^{(r)}X^{(p)}X^{(q)}\right] = Q_{lr}Q_{pq} + Q_{lp}Q_{rq} + Q_{lq}Q_{rp}, \qquad (3.3)$$

in particular $\operatorname{Var}((X^{(l)})^2) = 2Q^2_{ll}$, $\operatorname{Var}(X^{(l)}X^{(p)}) = Q_{ll}Q_{pp} + Q^2_{lp}$.

The LMM estimator is built from the data in experiment $\mathcal{E}_1$, but designed for the block-wise parametric model $\mathcal{E}_2$. In $\mathcal{E}_2$, the L^2-orthogonality of (φ_{jk}) as well as that of (Φ_{jk}) imply (cf. Reiß [19] in the scalar case)

$$S_{jk} \sim \mathbf{N}(0, C_{jk}) \text{ independent for all } (j, k) \qquad (3.4)$$

with covariance matrix

$$C_{jk} = \Sigma^{kh} + \pi^2 j^2 h^{-2} \operatorname{diag}(H^{kh}_{n,l})^2_l, \Sigma^{kh} = \bar{\Sigma}_h(kh),\, H^{kh}_{n,l} = (\overline{H^2}_{n,l,h}(kh))^{1/2}. \quad (3.5)$$

In the multivariate central limit theorem we are facing covariances between entries of the covariation matrix estimator, which we shall formalise by interpreting matrices as vectors: for a matrix $A \in \mathbb{R}^{d \times d}$ we consider the vector of its entries

$$vec(A) := \left(A_{11}, A_{21}, \ldots, A_{d1}, A_{12}, A_{22}, \ldots, A_{d2}, \ldots, A_{d(d-1)}, A_{dd} \right)^\top \in \mathbb{R}^{d^2}.$$

The natural estimator for $vec(C_{jk})$ is the empirical covariance $vec(S_{jk}S_{jk}^\top)$. We employ Kronecker (tensor) product calculus, where $A \otimes B \in \mathbb{R}^{d^2 \times d^2}$ for $A, B \in \mathbb{R}^{d \times d}$ is given by

$$(A \otimes B)_{p(d-1)+q, p'(d-1)+q'} = A_{pp'} B_{qq'}, \quad p, q, p', q' = 1, \ldots, d.$$

We evaluate the covariance matrix of $vec(S_{jk}S_{jk}^\top)$ in model $\mathcal{E}_2$ by introducing

$$\mathcal{Z} = \mathbb{C}\mathrm{OV}(vec(ZZ^\top)) \text{ for } Z \sim \mathbf{N}(0, E_d) \tag{3.6}$$

and applying the rule $vec(ABC) = (C^\top \otimes A)vec(B)$, see e.g. Fackler [9]:

$$\mathbb{C}\mathrm{OV}_{\mathcal{E}_2}(vec(S_{jk}S_{jk}^\top)) = \mathbb{C}\mathrm{OV}(vec(C_{jk}^{1/2}ZZ^\top C_{jk}^{1/2})) = (C_{jk} \otimes C_{jk})\mathcal{Z}. \tag{3.7}$$

We have used the commutativity of $\mathcal{Z}$ with $(C_{jk} \otimes C_{jk})^{1/2} = (C_{jk}^{1/2} \otimes C_{jk}^{1/2})$, which is easily checked using the actual form of $\mathcal{Z}$ derived from (3.3)

$$\mathcal{Z}_{p(d-1)+q, p'(d-1)+q'} = (1 + \delta_{p,q})\delta_{\{p,q\},\{p',q'\}}, p, q, p', q' = 1, \ldots, d,$$

or the equivalent property $\mathcal{Z}vec(A) = vec(A + A^\top)$ for all $A \in \mathbb{R}^{d \times d}$. Let us further introduce the Fisher information-type matrices

$$I_{jk} = C_{jk}^{-1} \otimes C_{jk}^{-1}, \quad I_k = \sum_{j=1}^\infty I_{jk}, \quad j \geq 1, \, k = 0, \ldots, h^{-1} - 1.$$

Our local method of moments estimator with oracle weights $\mathrm{LMM}_{or}^{(n)}$ uses that on each block a natural second moment estimator of Σ^{kh} is given as a convex combination of the bias-corrected empirical covariances:

$$\mathrm{LMM}_{or}^{(n)} := \sum_{k=0}^{h^{-1}-1} h \sum_{j=1}^\infty W_{jk} vec\left(S_{jk}S_{jk}^\top - \tfrac{\pi^2 j^2}{h^2} \mathrm{diag}\left((H_{n,l}^{kh})^2\right)_{1 \leq l \leq d} \right). \tag{3.8}$$

The optimal weight matrices W_{jk} in the oracle case are obtained as

$$W_{jk} := I_k^{-1} I_{jk} \quad \in \mathbb{R}^{d^2 \times d^2}. \tag{3.9}$$

Note that C_{jk}, I_{jk}, I_k and W_{jk} all depend on $(n_l)_{1 \leq l \leq d}$ and h, which is omitted in the notation. Finally, observe that (3.5) and $\sum_j W_{jk} = E_{d^2}$ imply that $\mathrm{LMM}_{or}^{(n)}$ is unbiased under $\mathcal{E}_2$.

3.2. *Asymptotic properties of the estimators.* We formulate the main result that the oracle estimator (3.8) and also a fully adaptive version for the integrated volatility matrix satisfy central limit theorems.

THEOREM 3.2. *Let Assumptions 2.1-(α), 2.2(ii-α) and 2.2(iii-$\underline{\Sigma}$) with $\alpha > 1/2$ hold true for observations from model $\mathcal{E}_1$. The oracle estimator (3.8) yields a consistent estimator for $vec(\int_0^1 \Sigma(s)\,ds)$ as $n_{min} \to \infty$ and $h = h_0 n_{min}^{-1/2}$ with $h_0 \to \infty$. Moreover, if $n_{max} = \mathcal{O}(n_{min}^{2\alpha})$ and $h = \mathcal{O}(n_{max}^{-1/4})$, then a multivariate central limit theorem holds:*

$$\mathbf{I}_n^{1/2}\left(\mathrm{LMM}_{or}^{(n)} - vec\left(\int_0^1 \Sigma(s)\,ds\right)\right) \xrightarrow{\mathcal{L}} \mathbf{N}\left(0, \mathcal{Z}\right) \ \text{in } \mathcal{E}_1 \quad (3.10)$$

with $\mathcal{Z}$ from (3.6) and $\mathbf{I}_n^{-1} = \sum_{k=0}^{h^{-1}-1} h^2 I_k^{-1}$.

We give a typical illustration with convergence rate $n_{min}^{1/4}$ and asymptotic covariance matrix.

COROLLARY 3.3. *Under the assumptions of Theorem 3.2 suppose $n_{min}/n_p \to \nu_p \in (0,1]$ for $p = 1,\ldots,d$ and introduce $\mathcal{H}(t) = \mathrm{diag}(\eta_p \nu_p^{1/2} F_p'(t)^{-1/2})_p \in \mathbb{R}^{d \times d}$ and $\Sigma_{\mathcal{H}}^{1/2} := \mathcal{H}(\mathcal{H}^{-1}\Sigma\mathcal{H}^{-1})^{1/2}\mathcal{H}$. Then*

$$n_{min}^{1/4}\left(\mathrm{LMM}_{or}^{(n)} - vec\left(\int_0^1 \Sigma(s)\,ds\right)\right) \xrightarrow{\mathcal{L}} \mathbf{N}\left(0, \mathbf{I}^{-1}\mathcal{Z}\right) \ \text{in } \mathcal{E}_1 \quad (3.11)$$

with

$$\mathbf{I}^{-1} = 2\int_0^1 (\Sigma \otimes \Sigma_{\mathcal{H}}^{1/2} + \Sigma_{\mathcal{H}}^{1/2} \otimes \Sigma)(t)\,dt.$$

In particular, the entries satisfy for $p, q = 1,\ldots,d$

$$n_{min}^{1/4}\left((\mathrm{LMM}_{or}^{(n)})_{p(d-1)+q} - \int_0^1 \Sigma_{pq}(s)\,ds\right) \xrightarrow{\mathcal{L}} \quad (3.12)$$

$$\mathbf{N}\left(0, 2(1+\delta_{p,q})\int_0^1 (\Sigma_{pp}(\Sigma_{\mathcal{H}}^{1/2})_{qq} + \Sigma_{qq}(\Sigma_{\mathcal{H}}^{1/2})_{pp} + 2\Sigma_{pq}(\Sigma_{\mathcal{H}}^{1/2})_{pq})(t)\,dt\right).$$

The variance (3.12) will coincide with the lower bound obtained in Section 4 below. The local noise level in $\mathcal{H}(t)$ depends on the observational noise level η_p and the local sample size $\nu_p^{-1}F_p'(t)$, $p = 1,\ldots,d$, after normalisation by n_{min}. It is easy to see that in the case $n_{min}/n_p \to 0$ the asymptotic variance vanishes for all entries (p,q), $q = 1,\ldots,d$. In general, also all other asymptotic variances are then reduced because the estimator profits from

correlation. In Section 4.2 the effect of $\mathcal{H}$ on $\Sigma_{\mathcal{H}}$ is discussed and illustrated in Figure 1. We infer the structure of the asymptotic covariance matrix using block-wise diagonalisation in Appendix B.

For a feasible estimator the optimal weight matrices $W_{jk} = W_j(\Sigma^{kh})$ and the information-type matrices $I_{jk} = I_j(\Sigma^{kh})$ are estimated in a preliminary step from the same data. To reduce variability in the estimate, a coarser grid of r^{-1} equidistant intervals, $r/h \in \mathbb{N}$ is employed for $\hat{W}_{jk}$. As derived in Bibinger and Reiß [4] for supremum norm loss and extended to L^1-loss and Besov regularity using the L^1-modulus of continuity as in the case of wavelet estimators (Cor. 3.3.1 in Cohen [7]), a preliminary estimator $\hat{\Sigma}(t)$ of the instantaneous covolatility matrix $\Sigma(t)$ exists with

$$\|\hat{\Sigma} - \Sigma\|_{L^1} = \mathcal{O}_P\left(n_{min}^{-\alpha/(4\alpha+2)}\right) \tag{3.13}$$

for $\Sigma \in B_{1,\infty}^\alpha([0,1])$. For k with $kh \in [mr, (m+1)r)$ we set

$$\hat{W}_{jk} = W_j(\hat{\Sigma}^{mr}), \ \hat{I}_{jk} = I_j(\hat{\Sigma}^{kh}) \text{ with } \hat{\Sigma}^{mr} = \bar{\hat{\Sigma}}_r(mr), \ \hat{\Sigma}^{kh} = \bar{\hat{\Sigma}}_h(kh).$$

The quadratic covariation matrix estimator with adaptive weights is then given by

$$\mathrm{LMM}_{ad}^{(n)} = \sum_{k=0}^{h^{-1}-1} h \sum_{j=1}^\infty \hat{W}_{jk} \, vec\left(S_{jk}S_{jk}^\top - \frac{\pi^2 j^2}{h^2} \operatorname{diag}\left((H_{n,l}^{kh})^2\right)_{1 \le l \le d}\right). \tag{3.14}$$

We estimate the total covariance matrix via

$$\hat{\mathbf{I}}_n^{-1} = \sum_{k=0}^{h^{-1}-1} h^2 \left(\sum_{j=1}^\infty \hat{I}_{jk}\right)^{-1}. \tag{3.15}$$

For $j \to \infty$ the weights $W_j(\Sigma)$ and the matrices $I_j(\Sigma)$ decay like j^{-4} in norm, compare Lemma C.1 below, such that in practice a finite sum over frequencies j suffices. By a tight bound on the derivatives of $\Sigma \mapsto W_j(\Sigma)$ we show in Appendix C.4 the following general result.

THEOREM 3.4. *Suppose* $\Sigma \in B_{1,\infty}^\alpha([0,1])$ *for* $\alpha \in (1/2, 1]$ *satisfying* $\alpha/(2\alpha+1) > \log(n_{max})/\log(n_{min}) - 1$. *Choose* $h, r \to 0$ *such that* $h_0 = hn_{min}^{1/2} \asymp \log(n_{min})$ *and* $n_{min}^{-\alpha/(2\alpha+1)} \lesssim r \lesssim (n_{min}/n_{max})^{1/2}$, $h^{-1}, r^{-1}, r/h \in \mathbb{N}$. *If the pilot estimator* $\hat{\Sigma}$ *satisfies* (3.13), *then under the conditions of Theorem 3.2 the adaptive estimator* (3.14) *satisfies*

$$\hat{\mathbf{I}}_n^{1/2}\left(\mathrm{LMM}_{ad}^{(n)} - vec\left(\int_0^1 \Sigma(s)\,ds\right)\right) \xrightarrow{\mathcal{L}} \mathbf{N}\left(0, \mathcal{Z}\right), \tag{3.16}$$

with $\hat{\mathbf{I}}_n$ from (3.15).

   Moreover, Corollary 3.3 applies equally to the adaptive estimator (3.14).

Since the estimated $\hat{\mathbf{I}}_n$ appears in the CLT, we have obtained a feasible limit theorem and (asymptotic) inference statements are immediate.

Some assumptions of Theorem 3.4 are tighter than for the oracle estimator. To some extent this is for the sake of clarity. Here, we have restricted Assumption 2.2(ii-α) to the Besov-regular part. A generalisation of the pilot estimator to martingales seems feasible, but is non-standard and might require additional conditions. We have also proposed a rather concrete choice of h and r, less is used in the proof, see e.g. (C.3) below.

The lower bound for α in terms of the sample-size ratio n_{max}/n_{min} is due to bounding norms of (estimated) information-type matrices separately. For $\alpha = 1$ (bounded variation case) the restriction imposes n_{max} to be somewhat smaller than $n_{min}^{4/3}$. By the Sobolev embedding $B_{1,\infty}^1 \subseteq H^\beta$ for all $\beta < 1/2$ the restriction $n_{max} = \mathcal{O}(n_{min}^{1+\beta})$ from Theorem 2.4 is clearly also satisfied in this case.

It is not clear whether a more elaborate analysis can avoid these restrictions. Still, to the best of our knowledge, a feasible CLT for asymptotically separating sample sizes has not been obtained before. The inclusion of possible jumps in Σ by measuring regularity only in the Besov scale is already in the scalar case a significant improvement over Reiß [19].

4. Semiparametric efficiency.

4.1. *Semiparametric Cramér-Rao bound.* We shall derive an efficiency bound for the following basic case of observation model ($\mathcal{E}_1$):

$$dY_t = X_t \, dt + \frac{1}{\sqrt{n}} \, dW_t, \quad X_t = \int_0^t \Sigma(s)^{1/2} dB_s, \quad t \in [0,1], \quad (4.1)$$

where

$$\Sigma(t) = \Sigma_0(t) + \varepsilon \mathbb{H}(t), \quad \Sigma_0(t)^{1/2} = O(t)^\top \Lambda(t) O(t). \quad (4.2)$$

We assume $\Sigma_0(t)$ and $\mathbb{H}(t)$ to be known symmetric matrices, $O(t)$ orthogonal matrices, $\Lambda(t) = \mathrm{diag}(\lambda_1(t), \ldots, \lambda_d(t))$ diagonal and consider $\varepsilon \in [-1,1]$ as unknown parameter. Furthermore, we require Assumption 2.2(iii-$\underline{\Sigma}$) for all Σ. Finally, we impose throughout this section the regularity assumption that the matrix functions $O(t), \mathbb{H}(t), \Lambda(t)$ are continuously differentiable.

The key idea is to transform the observation of dY_t in such a manner that the white noise part remains invariant in law while for the central parameter

$\Sigma(t) = \Sigma_0(t)$ the process X is transformed to a process with independent coordinates and constant volatility. It turns out that this can only be achieved at the cost of an additional drift in the signal. The construction first rotates the observations via $O(t)$, which diagonalises $\Sigma_0(t)$, and then applies a coordinate-wise time-transformation, corrected by a multiplication term to ensure L^2-isometry such that the white noise remains law-invariant.

We introduce the coordinate-wise time changes by

$$r_i(t) = \frac{\int_0^t \lambda_i(s)ds}{\int_0^1 \lambda_i(s)ds} \text{ and } (T_r g)(t) := (g_1(r_1(t)), \ldots, g_d(r_d(t)))^\top$$

for $g = (g_1, \ldots, g_d) : \mathbb{R} \to \mathbb{R}^d$. Moreover, we set

$$\bar{\Lambda} := \int_0^1 \Lambda(s)ds, \quad R'(t) := \bar{\Lambda}^{-1}\Lambda(t) = \operatorname{diag}(r_1'(t), \ldots, r_d'(t)).$$

LEMMA 4.1.   *By transforming $d\bar{Y} = T_r^{-1}\mathcal{M}_{(R')^{-1/2}O}dY$, the observation model* (4.1), (4.2) *is equivalent to observing*

$$d\bar{Y}(t) = S(t)\,dt + \frac{1}{\sqrt{n}}d\bar{W}(t) \qquad with \tag{4.3}$$

$$S(t) = T_r^{-1}\Big((R')^{-1}\Big(\int_0^\cdot ((R')^{-\frac{1}{2}}O)'(s)X(s)\,ds + \int_0^\cdot (R'(s))^{-\frac{1}{2}}O(s)\,dX(s)\Big)\Big)(t)$$

for $t \in [0, 1]$. At $\varepsilon = 0$ the observation $d\bar{Y}(t)$ reduces to

$$\Big(\int_0^t T_r^{-1}((R')^{-1}((R')^{-1/2}O)'X)(s)\,ds + \bar{\Lambda}\bar{B}(t)\Big)dt + \frac{1}{\sqrt{n}}d\bar{W}(t). \tag{4.4}$$

Here $\bar{W}$ and $\bar{B}$ are Brownian motions obtained from W and B, respectively, via rotation and time shift, as defined in (D.1) *below.*

If we may forget in (4.4) the first term, which is a drift term with respect to the martingale part $\bar{\Lambda}\bar{B}(t)$, then the central observation is indeed a constant volatility model in white noise. The lemma is proved in Appendix D.1.

Let us introduce the multiplication operator $\mathcal{M}_A g := Ag$ and the integration operator

$$Ig(t) = -\int_t^1 g(s)\,ds \text{ and its adjoint } I^*g(t) = -\int_0^t g(s)\,ds.$$

The covariance operator $C_{n,\varepsilon}$ on $L^2([0,1], \mathbb{R}^d)$ obtained from observing the differential in (4.3) is then given by

$$C_{n,\varepsilon} = T_r^*\mathcal{M}_{(R')^{1/2}O}I^*\mathcal{M}_{\Sigma_0+\varepsilon\mathbb{H}}I\mathcal{M}_{O^\top(R')^{1/2}}T_r + n^{-1}\operatorname{Id}.$$

The covariance operator $Q_{n,\varepsilon}$ when omitting the drift part is given by

$$Q_{n,\varepsilon} = Q_{n,0} + \varepsilon I^* T_r^* \mathcal{M}_M T_r I \text{ with } M(t) := ((R')^{-1/2} O \mathbb{H} O^\top (R')^{-1/2})(t)$$

where for $\varepsilon = 0$ the one-dimensional Brownian motion covariance operator C_{BM} appears

$$Q_{n,0} = \text{diag}(\bar{\lambda}_{ii} C_{BM} + n^{-1} \text{Id})_{1 \leq i \leq d}, \quad C_{BM} = I^* I.$$

We set $\dot{C}_0 = (C_{n,\varepsilon} - C_{n,0})/\varepsilon$ and $\dot{Q}_0 = (Q_{n,\varepsilon} - Q_{n,0})/\varepsilon$.

Standard Fisher information calculations for the finite-dimensional Gaussian scale model, e.g. [18, Chapter 6.6], transfer one-to-one to the infinite-dimensional case of observing $\mathbf{N}(0, Q_{n,\varepsilon})$ and yield as Fisher information for the parameter ε at $\varepsilon = 0$ the value

$$I_n^Q = \frac{1}{2} \| Q_{n,0}^{-1/2} \dot{Q}_0 Q_{n,0}^{-1/2} \|_{HS}^2 \ ,$$

because $Q_{n,0}^{-1/2} Q_{n,\varepsilon} Q_{n,0}^{-1/2}$ is differentiable at $\varepsilon = 0$ in Hilbert-Schmidt norm. In Appendix D.2 we show by Hilbert-Schmidt calculus, the Feldman-Hajek Theorem and the Girsanov Theorem that the models with and without drift do not separate:

LEMMA 4.2.   *We have*

$$\limsup_{n \to \infty} \| Q_{n,0}^{-1/2} \dot{Q}_0 Q_{n,0}^{-1/2} - C_{n,0}^{-1/2} \dot{C}_0 C_{n,0}^{-1/2} \|_{HS} < \infty.$$

Lemma 4.2 implies that the drift only contributes the negligible order $\mathcal{O}(1) = o(\sqrt{n})$ to the Fisher information. By identifying the hardest parametric subproblem for observations $\mathbf{N}(0, Q_{n,\varepsilon})$ we thus establish in Appendix D.3 a semiparametric Cramér-Rao bound for estimating any linear functional of the covolatility matrix. Further classical asymptotic statements like the local asymptotic minimax theorem would require the LAN-property of the parametric subproblem.

THEOREM 4.3.   *For a continuous matrix-valued function* $A : [0, 1] \to \mathbb{R}^{d \times d}$ *consider the estimation of*

$$\vartheta := \int_0^1 \langle A(t), \Sigma(t) \rangle_{HS} \, dt = \int_0^1 \sum_{i,j=1}^d A_{ij}(t) \Sigma_{ij}(t) \, dt \in \mathbb{R}. \qquad (4.5)$$

Then a hardest parametric subproblem in model (4.1), (4.2) *is obtained for the perturbation of* Σ_0 *by*

$$\mathbb{H}^*(t) = (\Sigma_0 (A + A^\top) \Sigma_0^{1/2} + \Sigma_0^{1/2} (A + A^\top) \Sigma_0)(t).$$

There any estimator $\hat{\vartheta}_n$ of ϑ, which is asymptotically unbiased in the sense $\frac{d}{d\vartheta}(\mathbb{E}_\vartheta[\hat{\vartheta}_n] - \vartheta) \to 0$, satisfies as $n \to \infty$

$$\mathbb{V}\mathrm{ar}_{\varepsilon=0}(\hat{\vartheta}_n) \geq \frac{(2 + \mathcal{O}(1))}{\sqrt{n}} \int_0^1 \langle(\Sigma_0 \otimes \Sigma_0^{1/2} + \Sigma_0^{1/2} \otimes \Sigma_0)\mathcal{Z}vec(A), \mathcal{Z}vec(A)\rangle(t)dt.$$

4.2. *Discussion.* The Cramér-Rao bound of Theorem 4.3 coincides with the asymptotic variance obtained in Corollary 3.3 in the case $\mathcal{H}(t) = E_d$. For different, but constant in time noise levels $\mathcal{H}$, we can apply a rescaling argument and replace in the lower bound model $\Sigma(t)$ by $\mathcal{H}^{-1}\Sigma(t)\mathcal{H}^{-1}$ and $A(t)$ by $\mathcal{H}A(t)\mathcal{H}$. This gives the more general Cramér-Rao bound

$$\mathbb{V}\mathrm{ar}_{\varepsilon=0}(\hat{\vartheta}_n) \geq \frac{(2 + \mathcal{O}(1))}{\sqrt{n}} \int_0^1 \langle(\Sigma_0 \otimes \Sigma_{0,\mathcal{H}}^{1/2} + \Sigma_{0,\mathcal{H}}^{1/2} \otimes \Sigma_0)\mathcal{Z}vec(A), \mathcal{Z}vec(A)\rangle(t)dt$$

with $\Sigma_{0,\mathcal{H}}^{1/2} = \mathcal{H}(\mathcal{H}^{-1}\Sigma_0\mathcal{H}^{-1})^{1/2}\mathcal{H}$, as in Corollary 3.3. If $\mathcal{H}(t)$ depends on t, rescaling generates another drift term, but if it varies smoothly in t, we expect to obtain again a lower bound that matches the asymptotic variance of our estimator. Let us discuss the efficient asymptotic variance **AVAR** further, concentrating on the homogeneous case $\mathcal{H}(t) = E_d$.

The efficient asymptotic variance for estimating $\int_0^1 \Sigma_{pp}(t)\,dt$ is

$$\mathbf{AVAR}\left(\int_0^1 \Sigma_{pp}(t)\,dt\right) = 8\int_0^1 \Sigma_{pp}(t)(\Sigma^{1/2}(t))_{pp}\,dt.$$

For the asymptotic variance of estimating $\int_0^1 \Sigma_{pq}(t)\,dt$ we obtain

$$2\int_0^1\left((\Sigma^{1/2})_{pp}\Sigma_{qq} + (\Sigma^{1/2})_{qq}\Sigma_{pp} + 2(\Sigma^{1/2})_{pq}\Sigma_{pq}\right)(t)\,dt.$$

Let us calculate specific examples. First, in the case $d = 1$, $\Sigma = \sigma^2$ this simplifies to

$$\mathbf{AVAR}\left(\int_0^1 \sigma^2(t)\,dt\right) = 8\int_0^1 \sigma^3(t)\,dt,$$

which agrees with the efficiency bound in Reiß [19]. For $p \neq q$ in the independent case $\Sigma = \mathrm{diag}(\sigma_p^2)_{1 \leq p \leq d}$ we find

$$\mathbf{AVAR}\left(\int_0^1 \Sigma_{pq}(t)\,dt\right) = 2\int_0^1 (\sigma_p^2\sigma_q + \sigma_p\sigma_q^2)(t)\,dt.$$

In the case $d = 2$ with spot volatilities $\sigma_1^2(t) = \sigma_2^2(t) = \sigma^2(t)$ and general correlation $\rho(t)$, i.e. $\sigma_{12}(t) = (\rho\sigma_1\sigma_2)(t)$, we obtain

$$\mathbf{AVAR}\left(\int_0^1 \sigma_1^2(t)\,dt\right) = 4\int_0^1 \sigma^3(t)\left(\sqrt{1 + \rho(t)} + \sqrt{1 - \rho(t)}\right)dt,$$

$$\mathbf{AVAR}\left(\int_0^1 \sigma_{12}(t)\,dt\right) = 2\int_0^1 \sigma^3(t)((1 + \rho(t))^{3/2} + (1 - \rho(t))^{3/2})\,dt.$$
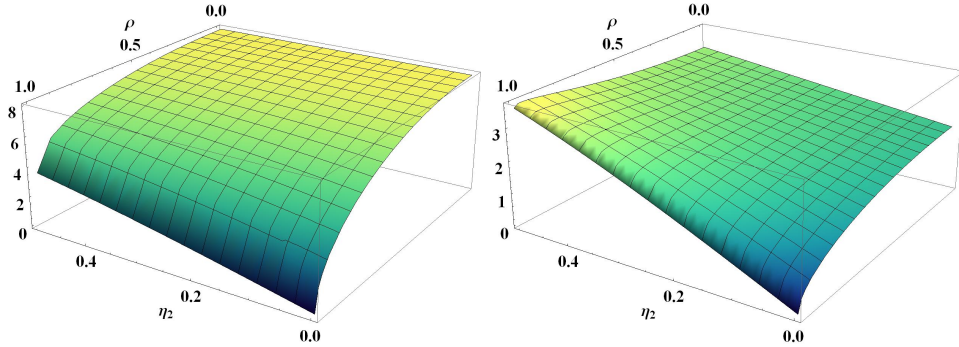
FIG 1. *Asymptotic variances of LMM for volatility σ_1^2 (left) and covolatility σ_{12} (right) plotted against correlation ρ and noise level η_2 (constant in time).*

With time-constant parameters these bounds decay for σ_1^2 (resp. grow for σ_{12}) in $|\rho|$ from $8\sigma^3$ (resp. $4\sigma^3$) at $\rho = 0$ to $4\sqrt{2}\sigma^3$ at $|\rho| = 1$ for both cases.

All the preceding examples can be worked out for different noise levels in $\mathcal{H}$, let us just highlight the bound for $\mathbf{AVAR}(\int_0^1 \Sigma_{pq}(t)dt)$:

$$2 \int_0^1 \left((\Sigma_{\mathcal{H}}^{1/2})_{pp}\Sigma_{qq} + (\Sigma_{\mathcal{H}}^{1/2})_{qq}\Sigma_{pp} + 2(\Sigma_{\mathcal{H}}^{1/2})_{pq}\Sigma_{pq} \right)(t) \, dt,$$

which matches the asymptotic variances obtained in Corollary 3.3. In general all noise levels enter for a fixed entry (p, q) via the matrix root $(\mathcal{H}^{-1}\Sigma\mathcal{H}^{-1})^{1/2}$ which only in the case of a diagonal covolatility matrix $\Sigma = \mathrm{diag}(\sigma_p^2)_p$ decouples as $\mathrm{diag}(\eta_p^{-1}\sigma_p)_p$ and where the bound simplifies to

$$p \neq q : \ 2 \int_0^1 \left(\eta_p\sigma_p\sigma_q^2 + \eta_q\sigma_q\sigma_p^2 \right)(t) \, dt; \quad p = q : \ 8 \int_0^1 \left(\eta_p\sigma_p^3 \right)(t) \, dt.$$

Figure 1 illustrates the general dependence of the asymptotic variance on the noise level via $\Sigma_{\mathcal{H}}$ in the case $d = 2$. The two volatilities are $\sigma_1 = \sigma_2 = 1$, the covolatility is $\sigma_{12} = \rho$ (constant in time) and the first noise level is $\eta_1 = 1$. The left plot shows the asymptotic variance of the estimator of σ_1^2 as a function of ρ and η_2. We see the significant gain of using observations from the other process for larger values of ρ. If the noise level η_2 for the second process is small, then the asymptotic variance can even approach zero. The plot on the right shows the same dependence for estimating the covolatility σ_{12}. For comparable size of η_2 and η_1 the asymptotic variance increases in ρ which is explained by the fact that also the value to be estimated increases. For small values of η_2, however, the efficiency gain by exploiting the correlation prevails.

For larger dimension d the variance can even be of order $\mathcal{O}(1/\sqrt{d})$; in the concrete case where all volatilities and noise levels equal 1, the asymptotic variance for estimating σ_1^2 can be reduced from 8 (using only observations from the first component or if Σ is diagonal) down to $8/\sqrt{d}$ (in case of perfect correlation).

We can also investigate the estimation of the entire quadratic covariation matrix $\int_0^1 \Sigma(t)\,dt$ and measure its loss with respect to the squared $d \times d$-Hilbert-Schmidt norm. Summing up the variances for each entry, we obtain the asymptotic risk

$$\frac{4}{\sqrt{n}}\int_0^1 \Big(\operatorname{trace}(\Sigma^{1/2})\operatorname{trace}(\Sigma) + \operatorname{trace}(\Sigma^{3/2})\Big)(t)\,dt.$$

This can be compared with the corresponding Hilbert-Schmidt norm error $\frac{1}{n}(\operatorname{trace}(\Sigma)^2 + \operatorname{trace}(\Sigma^2))$ for the empirical covariance matrix in an i.i.d. Gaussian $\mathbf{N}(0,\Sigma)$-setting.

For nonlinear functionals the Cramér-Rao bound is obtained by linearisation. Consider the prominent example of estimating power variations of the form $\int_0^1 (\Sigma_{pq})^{\rho/2}(t)dt$ for some $\rho > 0$ ($\rho = 4$ yields the so-called quarticity). Linearisation of the perturbation yields $\Sigma_{pq}^{\rho/2} = (\Sigma_0)_{pq}^{\rho/2} + \frac{\rho}{2}(\Sigma_0)_{pq}^{\rho/2-1}\varepsilon H_{pq} + \mathcal{O}(\varepsilon)$ provided $(\Sigma_0)_{pq} > 0$. We thus consider $A(t) = \frac{\rho}{2}(\Sigma_0(t))_{pq}^{\rho/2-1}(\delta_{(p,q),(p',q')})_{p',q'}$ and obtain the lower bound

$$\frac{\rho^2}{2}\int_0^1 (\Sigma_0)_{pq}^{\rho-2}\Big((\Sigma_0^{1/2})_{pp}(\Sigma_0)_{qq} + (\Sigma_0^{1/2})_{qq}(\Sigma_0)_{pp} + 2(\Sigma_0^{1/2})_{pq}(\Sigma_0)_{pq}\Big)(t)\,dt.$$

For $p = q$ this reduces to $2\rho^2 \int_0^1 ((\Sigma_0^{1/2})_{pp}(\Sigma_0)_{pp}^{\rho-1})(t)dt$, which is independent of Σ_0 only if $\rho = 1/2$ and Σ_0 is diagonal. In that case asymptotic equivalence with a homoskedastic Gaussian shift in terms of the mean function $\Sigma(t)^{1/4}$ holds, derived by independence of coordinates from Reiß [19], but for non-diagonal Σ_0 a variance-stabilising transform or even an equivalence result is not apparent.

5. Implementation and numerical results.

5.1. *Discrete-time estimator.* The construction to transfer discrete-time to continuous-time observations in the proof of Theorem 2.4 paves the way to the discrete approximation of the local spectral statistics (3.2). Using the interpolated process and integration by parts yields

$$\int \varphi_{jk}(t)dY^{(l)}(t) \asymp -\sum_{\nu=1}^{n_l} \int_{t_{\nu-1}^{(l)}}^{t_\nu^{(l)}} \Phi_{jk}(t)\frac{Y_\nu^{(l)} - Y_{\nu-1}^{(l)}}{t_\nu^{(l)} - t_{\nu-1}^{(l)}}\,dt.$$
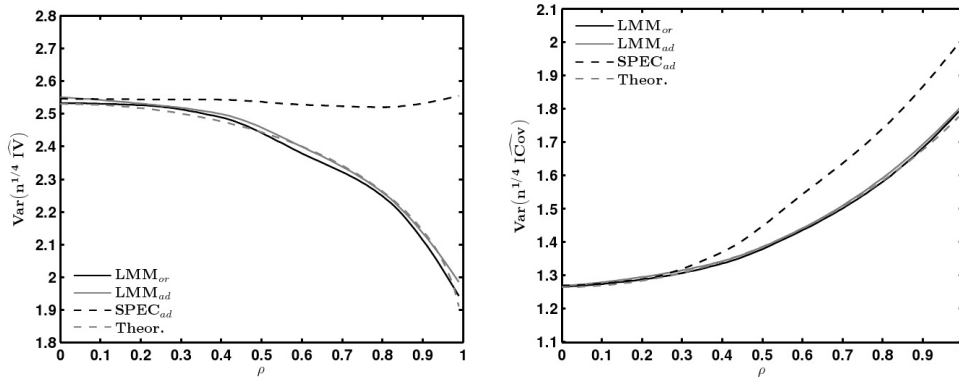
FIG 2. *Variances of estimators of σ_1^2 (left) and σ_{12} (right) in time-constant scenario $(n = 30,000)$.*

Hence, for discrete-time observations from $(\mathcal{E}_0)$ we use the local spectral statistics

$$S_{jk} = \pi j h^{-1} \Big(\sum_{\nu=1}^{n_l} \Big(Y_\nu^{(l)} - Y_{\nu-1}^{(l)} \Big) \Phi_{jk} \Big(\frac{t_{\nu-1}^{(l)} + t_\nu^{(l)}}{2} \Big) \Big)_{1 \le l \le d}. \qquad (5.1)$$

The noise terms in (3.5) translate from $\mathcal{E}_1$ to $\mathcal{E}_0$ via substituting $n_l^{-1} \int_{kh}^{(k+1)h} (F_l'(s))^{-1} \, ds$ by $\sum_{\nu: \, kh \le t_\nu^{(l)} \le (k+1)h} (t_\nu^{(l)} - t_{\nu-1}^{(l)})^2$. The discrete sum times h^{-1} can be understood as a block-wise quadratic variation of time in the spirit of Zhang *et al.* [20]. The bias is discretised analogously.

For the adaptive estimator we are in need of local estimates of $n_l F_l'$, Σ and estimators for $\eta_l^2, 1 \le l \le d$. It is well known how to estimate noise variances with faster $\sqrt{n_l}$-rates, see e.g. Zhang *et al.* [20]. Local observation densities can be estimated with block-wise quadratic variation of time as above, which then yield estimators $\hat{H}_{n,l}^{kh}$ of $H_{n,l}$ around time kh. Uniformly consistent estimators for $\Sigma(t), t \in [0, 1]$, are feasible, e. g. averaging spectral statistics for $j = 1, \dots, J$ over a set $\mathcal{K}_t$ of K adjacent blocks containing t:

$$\widehat{\Sigma}(t) = K^{-1} \sum_{k \in \mathcal{K}_t} J^{-1} \sum_{j=1}^{J} \Big(S_{jk} S_{jk}^\top - \pi^2 j^2 h^{-2} \operatorname{diag}((\hat{H}_{n,l}^{kh})_l^2) \Big) , \qquad (5.2)$$

We refer to Bibinger and Reiß [4] for details on the non-parametric pilot estimator with $J = 1$.

5.2. *Simulations.* We examine the finite-sample properties of the LMM for the case $d = 2$ in two scenarios. First, we compare the finite-sample vari-

ance with the asymptotic variances from Sections 3 and 4, i.e. for a parametric setup with $\eta_1^2 = \eta_2^2 = 0.1$, $\sigma_1 = \sigma_2 = 1$ and constant correlation ρ. We simulate $n_1 = n_2 = 30,000$ synchronous observations on $[0, 1]$. For estimating σ_1^2 and $\sigma_{12} = \rho$, Figure 2 displays the rescaled Monte-Carlo variance based on $20,000$ replications of the oracle and adaptive LMM (LMM$_{or}$ and LMM$_{ad}$), as well as the adaptive spectral estimator (SPEC$_{ad}$) by Bibinger and Reiß [4], which relies on the same spectral approach, but uses only scalar weighting instead of the full information matrix approach.

In practice the pilot estimator from (5.2) for J not too large performed well. As configuration we use $h^{-1} = 10$, $J = 30$ and $K = 8$, which turned out to be an accurate choice, but the estimators are reasonably robust to alternative input choices. For the LMM of σ_1^2, we observe the already familiar variance reduction effect associated with a growing signal correlation ρ, while the simulation-based variances of both LMM$_{or}$ and LMM$_{ad}$ are close to their theoretical asymptotic counterpart (Theor). The results for σ_{12} underline the precision gains compared to SPEC$_{ad}$ with univariate weights when ρ increases.

Next, we consider a complex and realistic stochastic volatility setting that relies on an extension of the widely-used Heston model, as e.g. employed by Aït-Sahalia *et al.* [1], accounting for both leverage effects and an intraday seasonality of volatility. The signal process for $l = 1, 2$ evolves as

$$dX_t^{(l)} = \varphi_l(t) \, \sigma_l(t) \, dZ_t^{(l)}, \qquad d\sigma_l^2(t) = \alpha_l \left(\mu_l - \sigma_l^2(t) \right) dt + \psi_l \, \sigma_l(t) \, dV_t^{(l)},$$

where $Z_t^{(l)}$ and $V_t^{(l)}$ are standard Brownian motions with $dZ_t^{(1)} dZ_t^{(2)} = \rho \, dt$ and $dZ_t^{(l)} dV_t^{(m)} = \delta_{l,m} \, \gamma_l \, dt$. $\varphi_l(t)$ is a non-stochastic seasonal factor with $\int_0^1 \varphi_l^2(t) \, dt = 1$. The unit time interval can represent one trading day, e.g. 6.5 hours or 23,400 seconds at NYSE.

We initialise the variance process $\sigma_l^2(t)$ by sampling from its stationary distribution $\Gamma \left(2 \, \alpha_l \, \mu_l / \psi_l^2, \psi_l^2 / (2\alpha_l) \right)$ and vary the value of the instantaneous signal correlation ρ, while setting $(\mu_l, \alpha_l, \psi_l, \gamma_l) = (1, 6, 0.3, -0.3)$, $l = 1, 2$, which under the stationary distribution, implies $\mathbb{E} \left[\int_0^1 \varphi_l^2(t) \, \sigma_l^2(t) \, dt \right] = 1$. The seasonal factor $\varphi_l(t)$ is specified in terms of intraday volatility functions estimated for S&P 500 equity data by the procedure in Anderson and Bollerslev [2]. $\varphi_1(t)$ and $\varphi_2(t)$ are based on cross-sectional averages of the 50 most and 50 least liquid stocks, respectively, which yields a pronounced L-shape in both cases (see Figure 3). We add noise processes that are i.i.d. $\mathbf{N}(0, \eta_l^2)$ and mutually independent with $\eta_l = 0.1 (\mathbb{E} \left[\int_0^1 \varphi_l^4(t) \, \sigma_l^4(t) \, dt \right])^{1/4}$, computed under the stationary distribution of $\sigma_l^2(t)$. Finally, asynchronicity effects are introduced by drawing observation times $t_i^{(l)}$, $1 \le i \le n_l$, $l = 1, 2$, from two
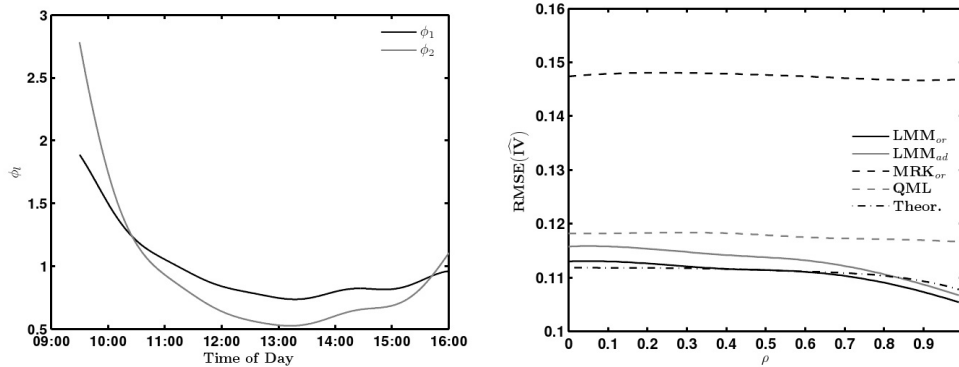
FIG 3. *Non-stochastic volatility seasonality factors (left) and RMSE for estimators of $\int_0^1 \varphi_1^2(t)\,\sigma_1^2(t)\,dt$ (right) in stochastic volatility scenario.*

independent Poisson processes with intensities $\lambda_1 = 1$ and $\lambda_2 = 2/3$ such that, on average, $n_1 = 23,400$ and $n_2 = 15,600$.

As a representative example, Figure 3 depicts the root mean-squared errors (RMSEs) based on $40,000$ replications of the following estimators of $\int_0^1 \varphi_1^2(t)\,\sigma_1^2(t)\,dt$: the oracle and adaptive LMM using $h^{-1} = 20$, $J = 15$ and $K = 8$, the quasi-maximum likelihood (QML) estimator by Aït-Sahalia *et. al.* [1] as well as an oracle version of the widely-used multivariate realised kernel (MRK$_{or}$) by Barndorff-Nielsen *et al.* [3]. For the latter, we employ the average univariate mean-squared error optimal bandwidth based on the true value of $\int_0^1 \varphi_l^4(t)\,\sigma_l^4(t)\,dt$, $l = 1, 2$. Further, we include the theoretical variance from the asymptotic theory (Theor), which is computed as the variance (3.12) averaged across all replications.

Three major results emerge. First, the LMM offers considerable precision gains when compared to both benchmarks. Second, a rising instantaneous signal correlation ρ is associated with a declining RMSE of the LMM, which is due to the decreasing variance, and thus confirms the findings from Section 3 in a realistic setting. Finally, the adaptive LMM closely tracks its oracle counterpart.

In summary, the simulation results show that the estimator has promising properties even in settings which are more general than those assumed in $(\mathcal{E}_1)$, allowing, for instance, for random observation times, stochastic intraday volatility as well as leverage effects. Even if the latter effects are not yet covered by our theory, the proposed estimator seems to be quite robust to deviations from the idealised setting.

APPENDIX A: FROM DISCRETE TO CONTINUOUS EXPERIMENTS

PROOF OF THEOREM 2.4. To establish Le Cam equivalence, we give a constructive proof to transfer observations in $\mathcal{E}_0$ to the continuous-time model $\mathcal{E}_1$ and the other way round. We bound the Le Cam distance by estimates for the squared Hellinger distance between Gaussian measures and refer to Section **A.1** in [19] for information on Hellinger distances between Gaussian measures and bounds with the Hilbert-Schmidt norm. The crucial difference here is that linear interpolation is carried out for non-synchronous irregular observation schemes. Consider the linear B-splines or hat functions

$$b_{i,n}(t) = \mathbb{1}_{\left[\frac{i-1}{n},\,\frac{i+1}{n}\right]}(t) \min\left(1 + n\left(t - \frac{i}{n}\right),\, 1 - n\left(t - \frac{i}{n}\right)\right).$$

Define $b_i^l(t) := b_{i,n_l}(F_l(t)), 1 \le i \le n_l, 1 \le l \le d$, which are warped spline functions satisfying $b_{i_1}^l(t_{i_2}^{(l)}) = \delta_{i_1,i_2}$. A centered Gaussian process $\hat{Y}$ is derived from linearly interpolating each component of Y:

$$\hat{Y}_t^{(l)} = \sum_{i=1}^{n_l} Y_i^{(l)} b_i^l(t) = \sum_{i=1}^{n_l} X_{t_i^{(l)}}^{(l)} b_i^l(t) + \sum_{i=1}^{n_l} \varepsilon_i^{(l)} b_i^l(t). \qquad (A.1)$$

The covariance matrix function $\mathbb{E}\left[\hat{Y}_t \hat{Y}_s^\top\right]$ of the interpolated process $\hat{Y}$ is determined by

$$\mathbb{E}\left[\hat{Y}_t^{(l)} \hat{Y}_s^{(r)}\right] = \sum_{i=1}^{n_l} \sum_{\nu=1}^{n_r} a_{lr}(t_i^{(l)} \wedge t_\nu^{(r)}) b_i^l(t) b_\nu^r(s) + \delta_{l,r} \eta_l^2 \sum_{i=1}^{n_l} b_i^l(t) b_i^l(s)$$

$$\text{with } A(t) = (a_{lr}(t))_{l,r=1,\dots,d} = \int_0^t \Sigma(s)\, ds\,.$$

For any $g = (g^{(1)}, \dots, g^{(d)})^\top \in L^2([0,1], \mathbb{R}^d)$ we have

$$\mathbb{E}\left[\langle \mathbf{g}, \hat{Y} \rangle^2\right] = \mathbb{E}\left[\left(\sum_{\nu=1}^d \langle g^{(v)}, \hat{Y}^{(v)} \rangle\right)^2\right]$$

$$= \sum_{l,r=1}^d \sum_{i=1}^{n_l} \sum_{\nu=1}^{n_r} a_{lr}(t_i^{(l)} \wedge t_\nu^{(r)}) \langle g^{(l)}, b_i^l \rangle \langle g^{(r)}, b_\nu^r \rangle + \sum_{l=1}^d \sum_{i=1}^{n_l} \langle g^{(l)}, b_i^l \rangle^2 \eta_l^2.$$

The sum of the addends induced by the observation noise in diagonal terms is bounded from above by

$$\sum_{l=1}^d \frac{\eta_l^2}{n_l} \left\|g^{(l)}/\sqrt{F_l'}\right\|_{L^2}^2 = \sum_{l=1}^d \left\|g^{(l)} H_{n,l}\right\|_{L^2}^2,$$

since by virtue of $0 \leq \sum_i b_{i,n} \leq 1$, $\int b_{i,n} = 1/n$ and Jensen's inequality:

$$
\begin{aligned}
\sum_{i=1}^{n_l} \langle g^{(l)}, b_i^l \rangle^2 &\leq \frac{1}{n_l} \sum_{i=1}^{n_l} \int_0^1 ((g^{(l)} \circ F_l^{-1}) \cdot (F_l^{-1})')^2 b_{i,n_l} \\
&\leq \frac{1}{n_l} \int_0^1 \left((g^{(l)} \circ F_l^{-1}) \cdot (F_l^{-1})' \right)^2 = \frac{1}{n_l} \int_0^1 \frac{(g^{(l)})^2}{F_l'} \,.
\end{aligned}
$$

On the other hand, we have

$$
\mathbb{E}[\langle g, \mathrm{diag}(H_{n,l})_l \, dW \rangle] = \sum_{l=1}^d \|g^{(l)} H_{n,l}\|_{L^2}^2
$$

for a d-dimensional standard Brownian motion W. Consequently, a process $\bar{Y}$ with continuous-time white noise and the same signal part as $\hat{Y}$ can be obtained by adding uninformative noise. Introduce the process

$$
d\bar{Y} = \Big(\sum_{i=1}^{n_l} X_{t_i^{(l)}} b_i^l(t) \Big)_{1 \leq l \leq d} dt + \mathrm{diag}(H_{n,l}(t))_{1 \leq l \leq d} \, dW_t \,, \quad \text{(A.2)}
$$

and its associated covariance operator $\bar{C} : L^2 \to L^2$, given by

$$
\bar{C}g(t) = \Big(\sum_{r=1}^d \sum_{i=1}^{n_l} \sum_{\nu=1}^{n_r} a_{lr}(t_i^{(l)} \wedge t_\nu^{(r)}) \langle g^{(r)}, b_\nu^r \rangle \Big)_{1 \leq l \leq d} + \left(H_{n,l}(t)^2 g^{(l)}(t) \right)_{1 \leq l \leq d} \,.
$$

In fact, it is possible to transfer observations from our original experiment $\mathcal{E}_0$ to observations of (A.2) by adding $\mathbf{N}\left(0, \bar{C} - \hat{C} \right)$-noise, where $\hat{C} : L^2 \to L^2$ is the covariance operator of $\hat{Y}$. Now, consider the covariance operator

$$
Cg(t) = \int_0^1 \left(\int_0^{t \wedge u} A(s) \, ds \right) g(u) \, du + \left(\frac{\eta_l^2}{n_l F_l'(t)} g^{(l)}(t) \right)_{1 \leq l \leq d} \,,
$$

associated with the continuous-time experiment $\mathcal{E}_1$.
We can bound $C^{-1/2}$ on $L^2([0,1], \mathbb{R}^d)$ from below (by partial ordering of operators) by a simple matrix multiplication operator:

$$
C^{-1/2} \leq \mathcal{M}_{\mathrm{diag}(H_{n,l}(t))_l}.
$$

Denote the Hilbert-Schmidt norm by $\| \cdot \|_{\mathrm{HS}}$. The asymptotic equivalence of observing $\bar{Y}$ and Y in $\mathcal{E}_1$ is ensured by the Hellinger distance bound

$$
\mathrm{H}^2 \left(\mathcal{L}\left(\bar{Y} \right), \mathcal{L}\left(Y \right) \right) \leq 2 \| C^{-1/2} \left(\bar{C} - C \right) C^{-1/2} \|_{\mathrm{HS}}^2
$$

$$\leq 2 \int_0^1 \int_0^1 \left(\sum_{l=1}^d \sum_{r=1}^d H_{n,l}(t)^{-2} H_{n,r}(t)^{-2} \right.$$

$$\left(\sum_{i=1}^{n_l} \sum_{\nu=1}^{n_r} a_{lr}(t_i^{(l)} \wedge t_\nu^{(r)}) b_i^l(t) b_\nu^r(s) - a_{lr}(t \wedge s) \right)^2 \Bigg) dt \, ds$$

$$= 2 \int_0^1 \int_0^1 \left(\sum_{l=1}^d \sum_{r=1}^d \frac{n_l n_r}{\eta_l^2 \eta_r^2} \right.$$

$$\left(\sum_{i=1}^{n_l} \sum_{\nu=1}^{n_r} a_{lr}(t_i^{(l)} \wedge t_\nu^{(r)}) b_{i,n_l}(u) b_{\nu,n_r}(z) - a_{lr}(F_l^{-1}(u) \wedge F_r^{-1}(z)) \right)^2 \Bigg) du \, dz$$

$$= \mathcal{O}\left(R^4 \sum_{l=1}^d \sum_{r=1}^d \eta_l^{-2} \eta_r^{-2} n_l n_r n_{min}^{-2-2\beta} \right).$$

The estimate for the L^2-distance between the function $(t,s) \mapsto A(F_l^{-1}(t) \wedge F_r^{-1}(s))$, $(l,r) \in \{1, \dots, d\}^2$, and its coordinate-wise linear interpolation by $\mathcal{O}(n_{min}^{-1-\beta} \vee n_{min}^{-3/2})$ relies on a standard approximation result on a rectangular grid of maximal width $(n_{min})^{-1}$ based on the fact that this function lies in the Sobolev class $H^{1+\beta}([0,1]^2)$ with corresponding norm bounded by $2R^4$. This follows immediately by the product rule from $A' = \Sigma \in H^\beta$ and $(F_l^{-1})' \in C^\beta$, together with an L^2-error bound at the skewed diagonal $\{(t,s) : F_l(t) = F_r(s)\}$.

Next, we explicitly show that $\mathcal{E}_1$ is at least as informative as $\mathcal{E}_0$. To this end we discretise in each component on the intervals $I_{i,l} = [\frac{i}{n_l} - \frac{1}{2n_l}, \frac{i}{n_l} + \frac{1}{2n_l}] \cap [0,1]$ for $i = 0, \dots, n_l$. Define

$$(Y_i')^{(l)} = \frac{1}{|I_{i,l}|} \int_{F_l^{-1}(I_{i,l})} F_l'(t) \, dY_t^{(l)} = \frac{1}{|I_{i,l}|} \int_{F_l^{-1}(I_{i,l})} X_t^{(l)} F_l'(t) \, dt + \varepsilon_i^{(l)}$$

$$= \frac{1}{|I_{i,l}|} \int_{I_{i,l}} X_{F^{-1}(u)}^{(l)} \, du + \varepsilon_i^{(l)}, \quad \text{(A.3)}$$

for $0 \leq i \leq n_l$ with i. i. d. random variables:

$$\varepsilon_i^{(l)} = \frac{1}{|I_{i,l}|} \int_{F_l^{-1}(I_{i,l})} \eta_l (F_l'/n_l)^{1/2} \, dW_t^{(l)} \overset{i.i.d.}{\sim} \mathbf{N}\left(0, \eta_l^2\right). \quad \text{(A.4)}$$

The covariances are calculated as

$$\mathbb{E}\left[(Y_i')^{(l)} (Y_\nu')^{(r)} \right] = \frac{1}{|I_{i,l}||I_{\nu,r}|} \int_{I_{i,l}} \int_{I_{\nu,r}} a_{lr}(F_l^{-1}(u) \wedge F_r^{-1}(u')) \, du \, du' + \delta_{l,r} \delta_{i,\nu} \eta_l^2.$$

We obtain for the squared Hellinger distance between the laws of observation

$$\mathrm{H}^2\left(\mathcal{L}\left((Y_i^{(l)})_{l=1,\dots,d; i=0,\dots,n_l} \right), \mathcal{L}\left(((Y_i')^{(l)})_{l=1,\dots,d; i=0,\dots,n_l} \right) \right)$$

$$\leq \sum_{l,r=1}^{d} \eta_l^{-2}\eta_r^{-2} \sum_{i=0}^{n_l} \sum_{\nu=0}^{n_r} \Big(\frac{1}{|I_{i,l}||I_{\nu,r}|} \int_{I_{i,l}} \int_{I_{\nu,r}} a_{lr}(F_l^{-1}(u) \wedge F_r^{-1}(u'))$$
$$- a_{lr}(F_l^{-1}(i/n_l \wedge \nu/n_r))\, du du' \Big)^2 .$$

Write $A_{lr}^F(u,u') = a_{lr}(F_l^{-1}(u) \wedge F_r^{-1}(u'))$ and note $A_{lr}^F \in H^{1+\beta}([0,1]^2)$ due to $A' = \Sigma \in H^\beta$ and $F_l^{-1}, F_r^{-1} \in C^\beta$. For $(i,\nu) \notin \mathcal{C} := \{(0,0),(0,n_r),(n_l,0),$ $(n_l,n_r)\}$ the rectangle $I_{i,l} \times I_{\nu,r}$ is symmetric around $(i/n_l, \nu/n_r)$ such that the integral in the preceding display equals (∇ denotes the gradient)

$$\int_{I_{i,l} \times I_{\nu,r}} \int_0^1 \Big(\big\langle \nabla A_{lr}^F \big(\tfrac{i}{n_l} + \vartheta \big(u - \tfrac{i}{n_l} \big), \tfrac{\nu}{n_r} + \vartheta \big(u' - \tfrac{\nu}{n_r} \big) \big), \big(u - \tfrac{i}{n_l}, u' - \tfrac{\nu}{n_r} \big) \big\rangle$$
$$- \big\langle \nabla A_{lr}^F \big(\tfrac{i}{n_l}, \tfrac{\nu}{n_r} \big), \big(u - \tfrac{i}{n_l}, u' - \tfrac{\nu}{n_r} \big) \big\rangle \Big)\, d\vartheta du du'.$$

Using Jensen's inequality we thus obtain further the bound for the squared Hellinger distance:

$$\sum_{l,r=1}^{d} \eta_l^{-2}\eta_r^{-2} \sum_{i=0}^{n_l} \sum_{\nu=0}^{n_r} \frac{(n_l \vee n_r)^{-2}}{|I_{i,l}||I_{\nu,r}|} \int_{I_{i,l} \times I_{\nu,r}} \int_0^1 \|\nabla A_{lr}^F(i/n_l + \vartheta(u - i/n_l),$$
$$\nu/n_r + \vartheta(u' - \nu/n_r)) - \nabla A_{lr}^F(i/n_l, \nu/n_r)\mathbb{1}((i,\nu) \notin \mathcal{C})\|^2\, d\vartheta\, du du'$$
$$= \sum_{l,r=1}^{d} \eta_l^{-2}\eta_r^{-2} \frac{n_l n_r}{(n_l \vee n_r)^2} \mathcal{O}\left(R^4 (n_l \wedge n_r)^{-2\beta}\right) = \mathcal{O}\left(R^4 \left(\sum_{l=1}^{d} n_l/\eta_l^2\right)^2 n_{min}^{-2-2\beta}\right)$$

where the order estimate is due to $\|\nabla A_{lr}^F\|_{H^\beta} \leq R^2$ and a standard L^2-approximation result for Sobolev spaces, observing that for the four corner rectangles in $\mathcal{C}$ the boundedness of the respective integrals only adds the total order $4n_{min}^{-2} < n_l n_r n_{min}^{-2-2\beta}$. □

APPENDIX B: ASYMPTOTICS IN THE BLOCK-WISE CONSTANT EXPERIMENT

PROOF OF THEOREM 3.2. As we have seen, the estimator is unbiased in $\mathcal{E}_2$. For the covariance structure we use the independence between blocks and frequencies and the commutativity with $\mathcal{Z}$ to infer

$$\mathbb{COV}_{\mathcal{E}_2}\big(\mathbf{I}_n^{1/2}\,\mathrm{LMM}_{or}^{(n)}\big) = \mathbf{I}_n^{1/2} \sum_{k=0}^{h^{-1}-1} h^2 \sum_{j=1}^{\infty} W_{jk} \mathbb{COV}_{\mathcal{E}_2}\left(vec\left(S_{jk}S_{jk}^\top\right)\right) W_{jk}^\top \mathbf{I}_n^{1/2}$$
$$= \mathbf{I}_n^{1/2} \sum_{k=0}^{h^{-1}-1} h^2 I_k^{-1} \mathbf{I}_n^{1/2} \mathcal{Z} = \mathcal{Z}. \tag{B.1}$$

Since the local Fisher-type informations are strictly positive definite and thus invertible by Assumption 2.2(*iii*), the multivariate CLT (3.10) for the oracle estimator follows by applying a standard CLT for triangular schemes as Theorem 4. 12 from [14]. The Lindeberg condition is implied by the stronger Lyapunov condition which is easily verified here by bounding moments of order 4.

In Appendix C below we prove that in experiment $\mathcal{E}_1$ the estimator $\text{LMM}_{or}^{(n)}$ has an additional bias of order $\mathcal{O}(n_{min}^{-\alpha/2}) + \mathcal{O}_P(h)$ and a difference in the covariance of order $\mathcal{O}(h n_{min}^{-\alpha/2}) + \mathcal{O}_P(h^2)$ under our Assumption 2.2(*ii-α*),(*iii-Σ*), which by Slutsky's lemma yields an asymptotically negligible term compared to the best attainable rate (in any entry) $n_{max}^{-1/4}$, cf. Theorem 4.3. □

PROOF OF COROLLARY 3.3. An important property of our oracle estimator is its equivariance with respect to invertible linear transformations A_k on each block k in the sense that for observed statistics $\tilde{S}_{jk} := A_k S_{jk} \sim \mathbf{N}(0, \tilde{C}_{jk})$ under $\mathcal{E}_2$ we obtain ($A^{-\top} := (A^\top)^{-1}$ for short)

$$C_{jk} = A_k^{-1}\tilde{C}_{jk}A_k^{-\top}, \; I_{jk} = (A_k \otimes A_k)^\top \tilde{I}_{jk}(A_k \otimes A_k), \; I_k = (A_k \otimes A_k)^\top \tilde{I}_k(A_k \otimes A_k)$$

and hence

$$\begin{aligned}
\text{LMM}_{or}^{(n)} &= \sum_{k=0}^{h^{-1}-1} h(A_k \otimes A_k)^{-1}\tilde{I}_k^{-1} \sum_{j\geq 0} \tilde{I}_{jk}(A_k \otimes A_k) vec(S_{jk}S_{jk}^\top) \\
&= \sum_{k=0}^{h^{-1}-1} (A_k \otimes A_k)^{-1}\Big(h\tilde{I}_k^{-1} \sum_{j\geq 0} \tilde{I}_{jk} vec(\tilde{S}_{jk}\tilde{S}_{jk}^\top)\Big).
\end{aligned}$$

For the covariance we use commutativity with $\mathcal{Z}$ and obtain likewise

$$\mathbb{COV}_{\mathcal{E}_2}(\text{LMM}_{or}^{(n)}) = \sum_{k=0}^{h^{-1}-1} h^2(A_k \otimes A_k)^{-1}\tilde{I}_k^{-1}(A_k \otimes A_k)^{-\top}\mathcal{Z}. \qquad \text{(B.2)}$$

We use this property to diagonalise the problem on each block. In terms of the noise level matrix $\mathcal{H}_k := \text{diag}(H_{l,n}^k)_{l=1,\ldots,d}$, let O_k be an orthogonal matrix such that

$$\Lambda^{kh} = O_k \mathcal{H}_k^{-1}\Sigma^{kh}\mathcal{H}_k^{-1}O_k^\top \qquad \text{(B.3)}$$

is diagonal. Note that Λ^{kh} grows with n, but we drop the dependence on n in the notation for all matrices Λ^{kh}, O_k and $\mathcal{H}_k$. Use $A_k = O_k \mathcal{H}_k^{-1}$ to obtain the spectral statistics (3.2) transformed:

$$\tilde{S}_{jk} = O_k \mathcal{H}_k^{-1}S_{jk} \sim \mathbf{N}\left(\mathbf{0}, \tilde{C}_{jk}\right) \text{ independent for all } (j, k)$$

which yields a simple-structured diagonal covariance matrix:

$$\tilde{C}_{jk} = O_k \mathcal{H}_k^{-1} C_{jk} \mathcal{H}_k^{-1} O_k^\top = \Lambda^{kh} + \frac{\pi^2 j^2}{h^2} E_d.$$

A key point is that the covariance structure (3.7) in $\mathbb{R}^{d^2 \times d^2}$ is for independent components $\tilde{S}_{jk}$ also diagonal, up to symmetry in the volatility matrix entries. Summing $\tilde{I}_{jk}$ over j is explicitly solvable and gives for $p, q = 1, \ldots, d$

$$(h\tilde{I}_k^{-1})_{p,q} = \Big(h^{-1} \sum_{j=1}^{\infty} (\tilde{C}_{jk}^{-1} \otimes \tilde{C}_{jk}^{-1})_{p,q}\Big)^{-1}$$

$$= \Big(h^{-1} \sum_{j=1}^{\infty} (\Lambda_{pp}^{kh} + \pi^2 j^2 h^{-2})^{-1} (\Lambda_{qq}^{kh} + \pi^2 j^2 h^{-2})^{-1}\Big)^{-1}$$

$$= \Big(\frac{\sqrt{\Lambda_{qq}^{kh}} \coth(h\sqrt{\Lambda_{pp}^{kh}}) - \sqrt{\Lambda_{pp}^{kh}} \coth(h\sqrt{\Lambda_{qq}^{kh}})}{2\sqrt{\Lambda_{pp}^{kh} \Lambda_{qq}^{kh}}(\Lambda_{qq}^{kh} - \Lambda_{pp}^{kh})} - \frac{1}{2h\Lambda_{pp}^{kh}\Lambda_{qq}^{kh}}\Big)^{-1}$$

$$= 2(\Lambda_{pp}^{kh}\sqrt{\Lambda_{qq}^{kh}} + \Lambda_{qq}^{kh}\sqrt{\Lambda_{pp}^{kh}})\Big(1 + \mathcal{O}\Big(e^{-2h\sqrt{\Lambda_{pp}^{kh} \wedge \Lambda_{qq}^{kh}}} + h^{-1}(\Lambda_{pp}^{kh} \wedge \Lambda_{qq}^{kh})^{-1/2}\Big)\Big),$$

using $\Lambda^{kh} \geq (\min_{l,t} n_l F_l'(t)\eta_l^{-2})\underline{\Sigma} \gtrsim n_{min}E_d$, $h^2 n_{min} \to \infty$ and $\coth(x) = 1 + \mathcal{O}(e^{-2x})$ for $x \to \infty$. We thus obtain uniformly over k

$$h\tilde{I}_k^{-1} = (2 + o(1))(\Lambda^{kh} \otimes \sqrt{\Lambda^{kh}} + \sqrt{\Lambda^{kh}} \otimes \Lambda^{kh}).$$

By formula (B.2) we infer in terms of $(\Sigma_{\mathcal{H}}^{kh})^{1/2} := \mathcal{H}_k(\mathcal{H}_k^{-1}\Sigma^{kh}\mathcal{H}_k^{-1})^{1/2}\mathcal{H}_k$

$$\mathbb{COV}_{\mathcal{E}_2}(\mathrm{LMM}_{or}^{(n)}) = (2 + o(1)) \sum_{k=0}^{h^{-1}-1} h(\Sigma^{kh} \otimes (\Sigma_{\mathcal{H}}^{kh})^{1/2} + (\Sigma_{\mathcal{H}}^{kh})^{1/2} \otimes \Sigma^{kh})\mathcal{Z}.$$

The final step consists in combining $n_{min}^{1/2} H_{n,l}(t) \to H_l(t)$ uniformly in t together with a Riemann sum approximation to conclude

$$\lim_{n_{min} \to \infty} n_{min}^{1/2} \mathbb{COV}_{\mathcal{E}_2}(\mathrm{LMM}_{or}^{(n)})$$

$$= 2\Big(\int_0^1 \Big(\Sigma \otimes (\mathcal{H}(\mathcal{H}^{-1}\Sigma\mathcal{H}^{-1})^{1/2}\mathcal{H}) + (\mathcal{H}(\mathcal{H}^{-1}\Sigma\mathcal{H}^{-1})^{1/2}\mathcal{H}) \otimes \Sigma\Big)(t)\, dt\Big)\mathcal{Z}.$$

$$\square$$

APPENDIX C: PROOFS FOR CONTINUOUS MODELS

C.1. Weight matrix estimates. We shall often need general norm bounds on the weight matrices W_{jk}.

LEMMA C.1.   *The oracle weight matrices satisfy uniformly over (j, k) and the matrices Σ^{kh} with $\|\Sigma^{kh}\|_\infty + \|(\Sigma^{kh})^{-1}\|_\infty \lesssim 1$*

$$\|W_{jk}\| \lesssim h_0^{-1}(1 + j^4/h_0^4)^{-1}.$$

PROOF. From the proof of Corollary 3.3 we infer

$W_{jk} = (H_k O_k^\top \otimes H_k O_k^\top)\tilde{W}_{jk}(O_k H_k^{-1} \otimes (O_k H_k^{-1})$ with
$\tilde{W}_{jk} = (2 + \mathcal{o}(1))h^{-1}\big((\Lambda^{kh}\tilde{C}_{jk}^{-1}) \otimes (\sqrt{\Lambda^{kh}}\tilde{C}_{jk}^{-1}) + (\sqrt{\Lambda^{kh}}\tilde{C}_{jk}^{-1}) \otimes (\Lambda^{kh}\tilde{C}_{jk}^{-1})\big).$

We evaluate one factor in W_{jk} using

$$\|H_k O_k^\top \Lambda^{kh}\tilde{C}_{jk}^{-1} O_k H_k^{-1}\| = \|\Sigma^{kh}(\Sigma^{kh} + \pi^2 j^2 h^{-2} H_k^2)^{-1}\| \lesssim (1 + j^2 h^{-2} n_{min}^{-2})^{-1}.$$

By $\|A \otimes B\| \leq \|A\|\|B\|$ and $\sqrt{\Lambda^{kh}}\tilde{C}_{jk}^{-1} = (\Lambda^{kh}\tilde{C}_{jk}^{-1})(\Lambda^{kh})^{-1/2}$ (the matrices are diagonal), we infer

$$\|W_{jk}\| \lesssim h^{-1}(1 + j^2 h_0^{-2})^{-2}\|H_k O_k^\top (\Lambda^{kh})^{-1/2} O_k H_k^{-1}\|.$$

To evaluate the last norm, despite matrix multiplication is non-commutative, we note

$$\left(O_k^\top (\Lambda^{kh})^{-\frac{1}{2}} O_k H_k^{-1}\right)^\top O_k^\top (\Lambda^{kh})^{-\frac{1}{2}} O_k H_k^{-1} = H_k^{-1} O_k^\top (\Lambda^{kh})^{-1} O_k H_k^{-1} = (\Sigma^{kh})^{-1},$$

whence by polar decomposition $|O_k^\top (\Lambda^{kh})^{-1/2} O_k H_k^{-1}| = (\Sigma^{kh})^{-1/2}$ implies

$$\|O_k^\top (\Lambda^{kh})^{-\frac{1}{2}} O_k H_k^{-1}\| = \|(\Sigma^{kh})^{-\frac{1}{2}}\| \lesssim 1.$$

Together with $\|H_k\| \lesssim n_{min}^{-1/2}$ this yields

$$\|W_{jk}\| \lesssim h^{-1}(1 + j^2 h_0^{-2})^{-2} n_{min}^{-1/2},$$

which gives the result.   □

Moreover, for the adaptive estimator we have to control the dependence of the weight matrices $W_{jk} = W_j(\Sigma^{kh})$ on Σ^{kh}. We use the notion of matrix differentiation as introduced in [9]: define the derivative dA/dB of a matrix-valued function $A(B) \in \mathbb{R}^{o \times p}$ with respect to $B \in \mathbb{R}^{q \times r}$ as the $\mathbb{R}^{op \times qr}$ matrix with row vectors $(d/dB_{ab})vec(A), 1 \leq a \leq q, 1 \leq b \leq r$.

LEMMA C.2.   *For the derivatives of the oracle weight matrices $W_j(\Sigma^{kh})$, assuming $\|\Sigma^{kh}\|_\infty + \|(\Sigma^{kh})^{-1}\|_\infty \lesssim 1$, we have uniformly over (j, k):*

$$\left\|\frac{d}{d\Sigma^{kh}}W_j(\Sigma^{kh})\right\| \lesssim h_0^{-1}(1 + j^4 h_0^{-4})^{-1}. \tag{C.1}$$

Proof. Since the notion of matrix derivatives relies on vectorisation, the identities $vec(I_k^{-1}I_{jk}) = (E_{d^2} \otimes I_k^{-1})vec(I_{jk}) = (I_{jk}^\top \otimes E_{d^2})vec(I_k^{-1})$ give rise to the matrix differentiation product rule

$$\frac{d}{d\Sigma^{kh}}W_{jk} = (I_{jk} \otimes E_{d^2})\frac{dI_k^{-1}}{d\Sigma^{kh}} + \left(E_{d^2} \otimes I_k^{-1}\right)\frac{dI_{jk}}{d\Sigma^{kh}}. \qquad (\text{C.2})$$

Applying the mixed product rule $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ repeatedly, and the differentiation product rule and chain rule to $I_{jk} = C_{jk}^{-1} \otimes C_{jk}^{-1}$, we obtain

$$\frac{d}{dC_{jk}}\left(C_{jk}^{-1} \otimes C_{jk}^{-1}\right) = -\left(\left(C_{jk}^{-1} \otimes C_{jk}^{-1}\right) \otimes \left(C_{jk}^{-1} \otimes C_{jk}^{-1}\right)\right)\left(\left((C_{jk} \otimes E_d \otimes E_{d^2})\right.\right.$$

$$+(E_{d^2} \otimes E_d \otimes C_{jk})\right)(E_d \otimes C_{d,d} \otimes E_d)\left((vec(E_d) \otimes E_{d^2}) + (E_{d^2} \otimes vec(E_d))\right)\right),$$

with the so-called commutation matrix $C_{d,d} = \mathcal{Z} - E_{d^2}$. By orthogonality of the last factors in both addends, $\|A \otimes B\| = \|A\|\|B\|$, and the mixed product rule, we infer for the norm of the second addend in (C.2)

$$\left\|(E_{d^2} \otimes I_k^{-1})\frac{dI_{jk}}{d\Sigma^{kh}}\right\| \leq 2 \left\|\left(E_d \otimes C_{jk}^{-1}\right) \otimes \left(I_k^{-1}(C_{jk}^{-1} \otimes C_{jk}^{-1})\right)\right\|$$

$$= 2 \|W_{jk}\| \left\|C_{jk}^{-1}\right\| \lesssim \|W_{jk}\|.$$

By virtue of

$$(I_k^{-1} \otimes E_{d^2})\frac{dI_k}{d\Sigma^{kh}} = -(E_{d^2} \otimes I_k)\frac{dI_k^{-1}}{d\Sigma^{kh}},$$

it follows with the mixed product rule that

$$dI_k^{-1}/d\Sigma^{kh} = -(I_k^{-1} \otimes I_k^{-1})(dI_k/d\Sigma^{kh}).$$

This yields for the norm of the first addend in (C.2)

$$\left\|(I_{jk} \otimes E_{d^2})\frac{dI_k^{-1}}{d\Sigma^{kh}}\right\| = \left\|(W_{jk}^\top \otimes I_k^{-1})\frac{dI_k}{d\Sigma^{kh}}\right\| \lesssim \|W_{jk}\| \left\|(E_{d^2} \otimes I_k^{-1})\sum_{j'}\frac{dI_{j'k}}{d\Sigma^{kh}}\right\|$$

$$\lesssim \|W_{jk}\|\Big(\sum_{j'}\|W_{j'k}\|\Big) \lesssim \|W_{jk}\|$$

since we can differentiate inside the sum by the absolute convergence of $\sum_{j'}\|W_{j'k}\|$. This proves our claim by Lemma C.1. $\qquad \square$

C.2. Bias bound. Using the formula $1 - 2\sin^2(x) = \cos(2x)$ and Itô isometry, the $(d \times d)$-matrix of (negative) biases (in the signal) of the addends in (3.8) as an estimator of Σ^{kh} in experiment $\mathcal{E}_1$ is given by

$$B_{j,k} := 2h^{-1} \int_{kh}^{(k+1)h} \Sigma(t) \cos(2j\pi h^{-1}(t - kh)) \, dt,$$

which has the structure of a j-th Fourier cosine coefficient. We introduce the corresponding weighting function in the time domain:

$$G_k(u) = 2 \sum_{j=1}^{\infty} W_{jk} \cos(2j\pi u) \in \mathbb{R}^{d^2 \times d^2}, \quad u \in [0, 1].$$

Parseval's identity then shows for the d^2-dimensional block-wise bias vector of (3.8):

$$\sum_{j=1}^{\infty} W_{jk} vec(B_{j,k}) = \int_{kh}^{(k+1)h} h^{-1} G_k(h^{-1}(t - kh)) vec(\Sigma(t)) \, dt.$$

The vector of total biases of (3.8) is then the linear functional of Σ:

$$\sum_{k=0}^{h^{-1}-1} h \sum_{j=1}^{\infty} W_{jk} vec(B_{jk}) = \int_0^1 G^h(t) vec(\Sigma(t)) \, dt$$

where for $t \in [kh, (k+1)h))$

$$G^h(t) = G_k(h^{-1}(t - kh)) = 2 \sum_{j=1}^{\infty} W_{jk} \cos(2\pi j h^{-1} t).$$

For Σ in the Besov space $B_{1,\infty}^{\alpha}([0,1])$, $0 < \alpha \le 1$, the L^1-modulus of continuity satisfies $\omega_{L^1([0,1])}(\Sigma, \delta) \le \|\Sigma\|_{B_{1,\infty}^{\alpha}} \delta^{\alpha}$, see e.g. [7, Section 3.2]. We have for $\delta \in (0, 1)$ and $s \in [0, 1 - \delta]$

$$\left| \int_0^{\delta} vec(\Sigma(t + s)) \cos(\tfrac{2\pi t}{\delta}) \, dt \right| = \frac{1}{\delta} \left| \int_0^{\delta} \int_0^{\delta} vec(\Sigma(t + s) - \Sigma(u + s)) \, du \right.$$

$$\left. \cos(\tfrac{2\pi t}{\delta}) \, dt \right| \le \sup_{0 \le v \le \delta} \int_0^{\delta} |vec(\Sigma(t + s) - \Sigma(t + v + s))| \, dt \le \omega_{L^1([s,s+\delta])}(\Sigma, \delta).$$

This shows for the total bias in estimation of the volatility in X by the bound on $\|W_{jk}\|$ in Lemma C.1

$$\left| \int_0^1 G^h(t) vec(\Sigma(t)) \, dt \right| \le 2 \sum_{k=0}^{h^{-1}-1} \sum_{j=1}^{\infty} \|W_{jk}\| \omega_{L^1([kh,(k+1)h])}(\Sigma, h/j)$$

$$\lesssim \sum_{j=1}^{\infty} h_0^{-1} (1 + (h_0/j)^4)^{-1} (h/j)^{\alpha} \asymp (h/h_0)^{\alpha} = n_{min}^{-\alpha/2}.$$

We thus have a bias of order $\mathcal{O}(n_{min}^{-\alpha/2})$. Remark that it is quite surprising that this bias bound is independent of h, which is also at the heart of the quasi-maximum likelihood method [1].

If $vec(\Sigma)$ is a (vector-valued) square-integrable martingale, then we use that martingale differences are uncorrelated and write for the total bias

$$\int_0^1 G^h(t)vec(\Sigma(t))\,dt = \int_0^1 G^h(t)vec(\Sigma(t) - \Sigma(\lfloor h^{-1}t\rfloor h))\,dt,$$

using $\int G_k = 0$. This expression is centred with covariance matrix

$$\sum_{k=0}^{h^{-1}-1} \int_{[kh,(k+1)h]^2} G_k(h^{-1}(t-kh))\mathbb{E}[vec(\Sigma(t)-\Sigma(kh))vec(\Sigma(s)-\Sigma(kh))^\top]$$
$$G_k(h^{-1}(s-kh))\,dtds.$$

The expected value in the display is smaller than (in matrix ordering) $\mathbb{E}[vec(\Sigma((k+1)h)-\Sigma(kh))vec(\Sigma((k+1)h)-\Sigma(kh))^\top]$. Because of $\|G_k\|_\infty \lesssim 1$ the covariance matrix (in any norm) is of order $\mathcal{O}(h^2\mathbb{E}[\|\Sigma(1)-\Sigma(0)\|^2]) = \mathcal{O}(h^2)$.

If $\Sigma = \Sigma^B + \Sigma^M$ is the sum of a function Σ^B in $B_{1,\infty}^\alpha([0,1])$ and a square-integrable martingale Σ^M, then the preceding estimations apply for each summand and the total bias has maximal order $\mathcal{O}(n_{min}^{-\alpha/2}) + \mathcal{O}_P(h)$.

C.3. Variance for general continuous-time model. The covariance for the estimator under model $\mathcal{E}_1$ can be calculated as under model $\mathcal{E}_2$, but we lose independence between different frequencies j, j' on the same block. For that we use the formula for Gaussian random vectors A, B

$$\mathbb{COV}(vec(AA^\top), vec(BB^\top)) = \big(\mathbb{COV}(B,B) \otimes \mathbb{COV}(A,B) + \mathbb{COV}(A,A)\otimes$$
$$\mathbb{COV}(A,B) + \mathbb{COV}(A,B) \otimes \mathbb{COV}(A,A) + \mathbb{COV}(A,B) \otimes \mathbb{COV}(B,B)\big)\mathcal{Z}/4,$$

obtained by polarisation. This implies

$$\|\mathbb{COV}_{\mathcal{E}_1}(\mathrm{LMM}_{or}^{(n)}) - \mathbb{COV}_{\mathcal{E}_2}(\mathrm{LMM}_{or}^{(n)})\|$$
$$\lesssim \sum_{k=0}^{h^{-1}-1} h^2 \sum_{j,j'=1}^\infty \|W_{j'k}\|\|W_{jk}(\mathbb{COV}_{\mathcal{E}_1}(S_{jk},S_{jk}) \otimes \mathbb{COV}_{\mathcal{E}_1}(S_{jk},S_{j'k}))\|.$$

From Lemma C.1 and $\|A \otimes B\| \leq \|A\|\|B\|$ for matrices A, B we infer that the series over j, j' is bounded in order by

$$\sum_{j,j'=1}^{\infty} h_0^{-2}(1 + j'/h_0)^{-4}(1 + j/h_0)^{-2} \Big(\Big\| \int_0^1 (\Sigma - \bar{\Sigma}_h)(t) \frac{\Phi_{jk}(t)\Phi_{j'k}(t)}{\|\Phi_{jk}\|_{L^2}\|\Phi_{j'k}\|_{L^2}} \, dt \Big\|$$
$$+ \Big\| \int_0^1 \mathrm{diag}(H_{n,l}^2 - \overline{H^2}_{n,l,h})(t) \, \varphi_{jk}(t)\varphi_{j'k}(t) \, dt \Big\| \Big).$$

The identities $2\cos(a)\cos(b) = \cos(a + b) + \cos(a - b)$, $2\sin(a)\sin(b) = \cos(a - b) - \cos(a + b)$ and the same bound as in Section C.2 imply for $\Sigma, (F_1')^{-1}, \ldots, (F_d')^{-1} \in B_{1,\infty}^{\alpha}([0,1])$ (note that even $(F_l')^{-1} \in C^{\alpha}([0,1])$)

$$\Big\| \int_0^1 (\Sigma - \bar{\Sigma}_h)(t) \frac{\Phi_{jk}(t)\Phi_{j'k}(t)}{\|\Phi_{jk}\|_{L^2}\|\Phi_{j'k}\|_{L^2}} \, dt \Big\| \lesssim h^{-1} \Big(\frac{h}{j + j'} + \frac{h(1 - \delta_{j,j'})}{|j - j'|} \Big)^{\alpha}$$
$$\times \|\Sigma\|_{B_{1,\infty}^{\alpha}([kh,(k+1)h])}$$

and similarly the bound

$$h^{-1} \Big(\frac{h}{j + j'} + \frac{h(1 - \delta_{j,j'})}{|j - j'|} \Big)^{\alpha} j j' h_0^{-2} \max_l \|(F_l')^{-1}\|_{B_{1,\infty}^{\alpha}([kh,(k+1)h])}$$

for the norm over $H_{n,l}^2$. Putting all estimates together gives

$$\|\mathbb{COV}_{\mathcal{E}_1}(\mathrm{LMM}_{or}^{(n)}) - \mathbb{COV}_{\mathcal{E}_2}(\mathrm{LMM}_{or}^{(n)})\|$$
$$\lesssim h \sum_{j,j'=1}^{\infty} h_0^{-2}(1 + j'/h_0)^{-4}(1 + j/h_0)^{-2} h^{\alpha}(1 + |j - j'|)^{-\alpha}(1 + jj'h_0^{-2}).$$

By comparison with the double integral (in terms of $x \approx j/h_0$, $y \approx j'/h_0$)

$$\int_0^{\infty} \int_0^{\infty} (1 + y)^{-4}(1 + x)^{-2}|x - y|^{-\alpha}(1 + xy) \, dx dy \lesssim 1$$

we conclude

$$\|\mathbb{COV}_{\mathcal{E}_1}(\mathrm{LMM}_{or}^{(n)}) - \mathbb{COV}_{\mathcal{E}_2}(\mathrm{LMM}_{or}^{(n)})\| \lesssim h n_{min}^{-\alpha/2}.$$

Arguing exactly as in Section C.2 for the case of Σ being a sum of a $B_{1,\infty}^{\alpha}$-function and an L^2-martingale, the difference of covariances is in general of order $\mathcal{O}(h n_{min}^{-\alpha/2}) + \mathcal{O}_P(h^2)$.

C.4. Proof of Theorem 3.4. Let us denote the rate of convergence of $\hat{\Sigma}$ by $\delta_n = n_{min}^{-\alpha/(4\alpha+2)}$. For later use we note the order bounds

$$\delta_n = o\Big(r^{1/2} h_0^{-1/2} (n_{min}/n_{max})^{1/4}\Big), \quad \delta_n = o\Big(h_0^{-1} (n_{min}/n_{max})^{1/2}\Big). \, \text{(C.3)}$$

First, we show that

$$\| \mathrm{LMM}_{or}^{(n)} - \mathrm{LMM}_{ad}^{(n)} \| = \mathcal{O}_P(n_{max}^{-1/4}) \tag{C.4}$$

which by Slutsky's Lemma implies the CLT with normalisation matrix $\mathbf{I_n}$. This in turn is already sufficient for obtaining the result of Corollary 3.3 for $\mathrm{LMM}_{ad}^{(n)}$. Let us start with proving that

$$T_n^m := \Bigg\| \sum_{m=0}^{r^{-1}-1} h \sum_{k=mr/h}^{(m+1)r/h-1} \sum_{j=1}^{\infty} \Big(W_j(\hat{\Sigma}^{mr}) - W_j(\Sigma^{mr})\Big) Z_{jk} \Bigg\| = \mathcal{O}_P(n_{max}^{-1/4})$$

where the random variables

$$Z_{jk} = vec\left(S_{jk} S_{jk}^{\top} - \pi^2 j^2 h^{-2} \operatorname{diag}\left((H_{n,l}^{kh})^2\right)_{1 \le l \le d} - \Sigma^{kh}\right)$$

are independent, $\mathbb{E}_{\mathcal{E}_2}[Z_{jk}] = 0$, $\mathbb{COV}_{\mathcal{E}_2}(Z_{jk}) = I_{jk}^{-1}\mathcal{Z}$. We have

$$T_n^m \le \sum_{m=0}^{r^{-1}-1} h \sum_{j=1}^{\infty} \Big\| W_j(\hat{\Sigma}^{mr}) - W_j(\Sigma^{mr}) \Big\| \Bigg\| \sum_{k=mr/h}^{(m+1)r/h-1} Z_{jk} \Bigg\|, \tag{C.5}$$

since the weight matrices do not depend on k on the same block of the coarse grid. Using Lemma C.2 and that $\|\hat{\Sigma} - \Sigma\|_{L^1} = \mathcal{O}_P(\delta_n)$, we obtain

$$\Big\| W_j(\hat{\Sigma}^{mr}) - W_j(\Sigma^{mr}) \Big\| \le \max_k \Big\| \frac{dW_j(\Sigma^{kh})}{d\Sigma^{kh}} \Big\| \|\hat{\Sigma}^{mr} - \Sigma^{mr}\|$$

$$= \mathcal{O}_P\left(\left(h_0^{-1} \wedge h_0^3 j^{-4}\right) r^{-1} \|\hat{\Sigma} - \Sigma\|_{L^1([mr,(m+1)r])}\right).$$

For the second factor in (C.5) we employ $\|\mathbb{COV}_{\mathcal{E}_2}(Z_{jk})\| = 2\|C_{jk}\|^2$ such that $\|Z_{jk}\| = \mathcal{O}_P(\|C_{jk}\|)$. Consequently, (C.3) implies for T_n^m the bound

$$\sum_{m=0}^{r^{-1}-1} h \sum_{j=1}^{\infty} \mathcal{O}_P\left((h_0^{-1} \wedge h_0^3 j^{-4}) r^{-1} \|\hat{\Sigma} - \Sigma\|_{L^1([mr,(m+1)r])} (r/h)^{1/2} (1 \vee j^2 h_0^{-2})\right)$$

$$= \mathcal{O}_P\left(r^{-1/2} h^{1/2} \delta_n\right) = \mathcal{O}_P(n_{max}^{-1/4}).$$

The asymptotics (C.4) follow if we can ensure that the coarse grid approximations of the weights induce a negligible error, i.e. if also

$$\sum_{m=0}^{r^{-1}-1} \sum_{k=mr/h}^{(m+1)r/h-1} h^2 \sum_{j=1}^{\infty} \left(W_j(\Sigma^{kh}) - W_j(\Sigma^{mr}) \right) Z_{jk} = \mathcal{O}_P(n_{max}^{-1/4})$$

holds. The term is centred and its covariance matrix is bounded in norm by

$$\sum_{m=0}^{r^{-1}-1} \sum_{k=mr/h}^{(m+1)r/h-1} h^2 \sum_{j=1}^{\infty} \left\| W_j(\Sigma^{kh}) - W_j(\Sigma^{mr}) \right\|^2 \|I_{jk}^{-1}\|.$$

From Lemma C.2, $\|I_{jk}^{-1}\| = 2\|C_{jk}\|^2 \lesssim 1 + j^4 h_0^{-4}$ and $\Sigma \in B_{1,\infty}^{\alpha}([0,1])$ we derive the upper bound

$$\mathcal{O}\Big(\sum_{k=0}^{h^{-1}-1} h^2 \sum_{j=1}^{\infty} r^2 h_0^{-2}(1 + j^4 h_0^{-4})^{-1} \Big) = \mathcal{O}\Big(n_{min}^{-1/2} r^{2\alpha} \Big) = o(n_{max}^{-1/2})$$

by the choice of r and $\alpha > 1/2$.

Another application of Slutsky's Lemma yields the CLT with normalisation matrix $\hat{\mathbf{I}}_n$ provided $\mathbf{I}_n^{1/2} \hat{\mathbf{I}}_n^{-1/2} \to E_{d^2}$ in probability. The proof of Lemma C.2, more specifically the bound on the last term in (C.2), yields also

$$\left\| \frac{d}{d\Sigma^{kh}} I_j(\Sigma^{kh}) \right\| \lesssim h_0^{-1}(1 + j^4 h_0^{-4})^{-1}.$$

This implies $\sum_{k,j} \|\hat{I}_{jk} - I_{jk}\| = \mathcal{O}_P(h^{-1}\delta_n)$. Using $\hat{A}^{-1} - A^{-1} = A^{-1}(\hat{A} - A)\hat{A}^{-1}$ and $\|I_k^{-1}\| \lesssim h_0^{-1}$, we infer

$$\|\hat{\mathbf{I}}_n^{-1} - \mathbf{I}_n^{-1}\| \le \sum_{k=0}^{h^{-1}-1} h^2 \left\| \Big(\sum_{j=1}^{\infty} \hat{I}_{jk} \Big)^{-1} - \Big(\sum_{j=1}^{\infty} I_{jk} \Big)^{-1} \right\| = \mathcal{O}_P\Big(h\delta_n h_0^{-2} \Big).$$

The smallest eigenvalue of $\mathbf{I}_n^{-1}$ equals $\|\mathbf{I}_n\|^{-1}$ which has order at least $n_{max}^{-1/2}$. The global Lipschitz constant L_n of $f(x) = x^{1/2}$ for $x \ge \|\mathbf{I}_n\|^{-1}$ is therefore of order $n_{max}^{1/4}$. The perturbation result from [16] for functional calculus therefore implies

$$\|\mathbf{I}_n^{1/2} \hat{\mathbf{I}}_n^{-1/2} - E_d\| \le L_n \|\mathbf{I}_n^{1/2}\| \|\mathbf{I}_n^{-1} - \hat{\mathbf{I}}_n^{-1}\| = \mathcal{O}_P\Big(n_{max}^{1/2} h\delta_n h_0^{-2} \Big).$$

The order is $(n_{max}/n_{min})^{1/2} h_0^{-1}\delta_n$ and tends to zero by (C.3).

APPENDIX D: PROOF OF THE LOWER BOUND

D.1. Proof of Lemma 4.1. Since $\mathcal{M}_{(R')^{1/2}}T_r$ is an isometry on $L^2([0,1]; \mathbb{R}^d)$, we obtain directly for the adjoint $T_r^* = T_r^{-1}\mathcal{M}_{(R')^{-1}}$. We observe in a formal differential notation:

$$
\begin{aligned}
T_r^*\mathcal{M}_{(R')^{1/2}O}dY &= T_r^{-1}\mathcal{M}_{(R')^{-1/2}O}(Xdt + \tfrac{1}{\sqrt{n}}dW) \\
&= -T_r^{-1}I^*(\mathcal{M}_{((R')^{-1/2}O)'}Xdt + \mathcal{M}_{(R')^{-1/2}O}dX) + \tfrac{1}{\sqrt{n}}d\bar{W} \\
&= -I^*T_r^*(\mathcal{M}_{((R')^{-1/2}O)'}Xdt + \mathcal{M}_{(R')^{-1/2}O}dX) + \tfrac{1}{\sqrt{n}}d\bar{W}.
\end{aligned}
$$

Here, we use that $T_r^*\mathcal{M}_{(R')^{1/2}O}$ is an L^2-isometry and we introduce the independent Brownian motions $\bar{W}, \bar{B}$ via the differentials

$$
d\bar{W} = T_r^*\mathcal{M}_{(R')^{1/2}O}\,dW, \quad d\bar{B} = T_r^*\mathcal{M}_{(R')^{1/2}O}\,dB
$$

or alternatively (apply $-I^*$) via their coordinates $i = 1, \ldots, d$ as

$$
\bar{W}_i(u) = \sum_{j=1}^d \int_0^{r_i^{-1}(u)} R'_{ii}(s)^{1/2}O_{ij}(s)\,dW_j(s), \tag{D.1}
$$

and $\bar{B}_i(u)$ analogously.

The formal derivations are made rigorous by duality, that is testing stochastic differentials with deterministic L^2-functions. We infer from the coordinate-wise definition of $\bar{W}$ for $f \in L^2([0,1]; \mathbb{R}^d)$ (e.g., check via indicator functions f)

$$
\int_0^1 \langle O(t)^\top R'(t)^{1/2}(T_r f)(t), dW_t \rangle = \int_0^1 \langle f(u), d\bar{W}(u) \rangle
$$

and equally for $\bar{B}$. Now consider for functions $g \in L^2([0,1]; \mathbb{R}^d)$ the real

observations

$$\int_0^1 \langle O(t)^\top R'(t)^{1/2}(T_r g)(t), dY_t \rangle = \int_0^1 \langle O(t)^\top R'(t)^{-1/2}(T_r Ig)'(t), dY_t \rangle$$

$$= \int_0^1 \langle (O(t)^\top R'(t)^{-1/2}(T_r Ig))' - (O(t)^\top R'(t)^{-1/2})'(T_r Ig)(t), X_t \rangle \, dt$$

$$+ \frac{1}{\sqrt{n}} \int_0^1 \langle O(t)^\top R'(t)^{1/2}(T_r g)(t), dW_t \rangle$$

$$= \int_0^1 \langle -(O(t)^\top R'(t)^{-1/2})'(T_r Ig)(t), X_t \rangle \, dt$$

$$- \int_0^1 \langle O(t)^\top R'(t)^{-1/2}(T_r Ig)(t), dX_t \rangle + \frac{1}{\sqrt{n}} \int_0^1 \langle g(u), d\bar{W}_u \rangle$$

$$= \int_0^1 \langle -(O(t)^\top R'(t)^{-1/2})'(T_r Ig)(t), X_t \rangle \, dt$$

$$- \int_0^1 \langle \Sigma(t)^{1/2}O(t)^\top R'(t)^{-1/2}(T_r Ig)(t), dB_t \rangle + \frac{1}{\sqrt{n}} \int_0^1 \langle g(u), d\bar{W}_u \rangle.$$

For $\varepsilon = 0$ we use $(R')^{-1/2}\Lambda(R')^{-1/2} = \bar{\Lambda}$ and evaluate the first two terms of the last display as

$$\int_0^1 \langle -(O(t)^\top R'(t)^{-1/2})'(T_r Ig)(t), X_t \rangle \, dt - \int_0^1 \langle (T_r^{-1}\bar{\Lambda}(T_r Ig))(u), d\bar{B}_u \rangle.$$

As $\bar{\Lambda}$ is constant in time, the second term is equal to $-\int_0^1 \langle Ig, \bar{\Lambda}d\bar{B} \rangle$ and the formal derivations above are confirmed.

D.2. Proof of Lemma 4.2. In a first step note that for general operators A, B we have

$$\|AA^* - BB^*\|_{HS}^2 = \tfrac{1}{2}\|(2A + B - A)(A - B)^* + (2A + B - A)^*(A - B)\|_{HS}$$

$$\leq 2\|A\|\|A - B\|_{HS} + \|A - B\|_{HS}^2.$$

Hence, it suffices to show

$$\|Q_{n,0}^{-1/2}Q_{n,1}^{1/2}\| \lesssim 1 \text{ and } \|Q_{n,0}^{-1/2}Q_{n,1}^{1/2} - C_{n,0}^{-1/2}C_{n,1}^{1/2}\|_{HS} \lesssim 1.$$

A further reduction is achieved by splitting terms to obtain

$$\|Q_{n,0}^{-1/2}Q_{n,1}^{1/2} - C_{n,0}^{-1/2}C_{n,1}^{1/2}\|_{HS} \leq \|\mathrm{Id} - C_{n,0}^{-1/2}Q_{n,0}^{1/2}\|_{HS}\|Q_{n,0}^{-1/2}Q_{n,1}^{1/2}\|$$

$$+ \|C_{n,0}^{-1/2}Q_{n,0}^{1/2}\|\|Q_{n,0}^{-1/2}Q_{n,1}^{1/2}\|\|\mathrm{Id} - Q_{n,1}^{-1/2}C_{n,1}^{1/2}\|_{HS}.$$

Owing to $\|C_{n,0}^{-1/2}Q_{n,0}^{1/2}\| \leq 1 + \|\mathrm{Id} - C_{n,0}^{-1/2}Q_{n,0}^{1/2}\|_{HS}$ it remains to show

$$\|Q_{n,0}^{-1/2}Q_{n,1}^{1/2}\| \lesssim 1, \quad \|\mathrm{Id} - C_{n,0}^{-1/2}Q_{n,0}^{1/2}\|_{HS} \lesssim 1 \text{ and } \|\mathrm{Id} - Q_{n,0}^{-1/2}C_{n,0}^{1/2}\|_{HS} \lesssim 1.$$

Finally, we can use $Q_{n,1} - Q_{n,0} = Q_{\infty,1} - Q_{\infty,0}$, $Q_{n,1} \geq Q_{\infty,1}$ in operator order (and similarly for $C_{n,\varepsilon}$) as well as $|a - 1| \leq |a^2 - 1|$ for $a \geq 0$ implying $\|A - \mathrm{Id}\|_{HS} \leq \|AA^* - \mathrm{Id}\|_{HS}$ for positive operators A. We are thus left with proving that the following three quantities are uniformly bounded

$$\|Q_{n,0}^{-1/2}Q_{n,1}^{1/2}\|, \|C_{\infty,0}^{-1/2}(Q_{\infty,0}-C_{\infty,0})C_{\infty,0}^{-1/2}\|_{HS}, \|Q_{\infty,1}^{-1/2}(C_{\infty,1}^{1/2}-Q_{\infty,1}^{1/2})Q_{\infty,1}^{-1/2}\|_{HS}.$$

By the Feldman-Hajek Theorem for Gaussian measures, see e.g. [8], the latter two quantities are finite iff the Gaussian laws $\mathbf{N}(0, C_{\infty,\varepsilon})$ and $\mathbf{N}(0, Q_{\infty,\varepsilon})$, are equivalent for $\varepsilon \in \{0, 1\}$. Using again differential notation, these are the laws of

$$Z^C := T_r^* \mathcal{M}_{(R')^{1/2}O} X, \quad Z^Q := -I^* T_r^* \mathcal{M}_{(R')^{-1/2}O} dX$$

where $dX = \Sigma^{1/2} dB$ for the ε at hand. Both processes are images in $C([0, 1], \mathbb{R}^d)$ under the linear (and thus measurable) map $T_r^{-1} = T_r^* \mathcal{M}_{R'}$ of the respective processes

$$\tilde{Z}^C := \mathcal{M}_{(R')^{-1/2}O} X, \quad \tilde{Z}^Q := -I^* \mathcal{M}_{(R')^{-1/2}O} dX.$$

By the product rule we see

$$\tilde{Z}^C(t) = -I^* \{\mathcal{M}_{(R')^{-1/2}O} dX + \mathcal{M}_{((R')^{-1/2}O)'} X\}(t)$$
$$= \tilde{Z}^Q(t) + \int_0^t ((R')^{-1/2}O)'(s)X(s)\,ds.$$

Hence, $\tilde{Z}^C$ equals the Brownian martingale $\tilde{Z}^Q$ plus an adapted linear drift in X. By Girsanov's theorem, noting that all deterministic quantities are continuous and bounded away from zero, the laws of $\tilde{Z}^C$ and $\tilde{Z}^Q$ are equivalent, e.g. use Thm. 3.5.1. together with Cor. 3.5.16 in [15]. Hence, so are the laws of their images Z^C and Z^Q, as required.

Let us finally consider $Q_{n,0}^{-1/2}Q_{n,1}^{1/2}$. Its squared norm equals

$$\sup_{f \in L^2} \frac{\langle Q_{n,1}f, f\rangle}{\langle Q_{n,0}f, f\rangle} = \sup_{f \in L^2} \frac{\langle \mathcal{M}_{(R')^{-1/2}O\Sigma_1 O^\top (R')^{-1/2}} T_r I f, T_r I f\rangle + \frac{1}{n}\|f\|^2}{\|If\|^2 + \frac{1}{n}\|f\|^2}$$
$$\leq (1 + \|M\|_\infty) \max_{i=1,\dots,d} \|(r_i^{-1})'\|_\infty.$$

This uniform bound is finite under our regularity assumptions.

D.3. Proof of Theorem 4.3. Without loss of generality we may assume that $A(t)$ is symmetric for all t because $\Sigma(t)$ is symmetric. Owing to $\mathcal{Z}vec(A) = 2vec(A)$ and $(\Sigma_0 \otimes \Sigma_0^{1/2})vec(A) = vec(\Sigma_0^{1/2}A\Sigma_0)$, we thus have to show in terms of the Hilbert-Schmidt scalar product

$$\mathbb{Var}_{\varepsilon=0}(\hat{\vartheta}_n) \geq \frac{(8 + \mathcal{O}(1))}{\sqrt{n}} \int_0^1 \langle ((\Sigma_0 \otimes \Sigma_0^{1/2} + \Sigma_0^{1/2} \otimes \Sigma_0)A)(t), A(t)\rangle_{HS}dt.$$

Since C_{BM} is a positive operator on $L^2([0,1];\mathbb{R})$, we can define the bounded self-adjoint operator

$$\Delta_n^\sigma = I(\sigma^2 C_{BM} + \frac{1}{n}\mathrm{Id})^{-1}I^* = (\sigma^2\,\mathrm{Id} + \frac{1}{n}C_{BM}^{-1})^{-1}.$$

C_{BM} is Hilbert-Schmidt and so is Δ_n^σ. We identify its kernel $\delta_n^\sigma : [0,1]^2 \to \mathbb{R}$ (or Green function) as

$$\delta_n^\sigma(t,s) = \frac{\sqrt{n}}{2\sigma\cosh(\sigma\sqrt{n})}\Big(\sinh(\sigma\sqrt{n}(1-|s-t|)) + \sinh(\sigma\sqrt{n}(t+s-1))\Big).$$

This can be formally derived from the properties $C_{BM}^{-1} = -D^2$ on its domain, δ_n^σ in the domain (i.e. $\delta_n^\sigma(0,s) = 0$, $(\delta_n^\sigma)'(1,s) = 0$) and $\delta_n^\sigma(t,s) = \Delta_n^\sigma\delta_s(t)$. Alternatively use the eigenvalue-eigenfunction decomposition of C_{BM} and apply functional calculus. The main observation is that δ_n^σ has all the properties of a smoothing kernel, which for $n \to \infty$ concentrates on the diagonal $\{t = s\}$, where it approximates the uniform law. This is best seen by the approximation for large n

$$\delta_n^\sigma(t,s) \asymp \frac{\sqrt{n}}{2\sigma}\Big(\exp(-\sigma\sqrt{n}|s-t|) + \mathrm{sgn}(t+s-1)\exp(\sigma\sqrt{n}(|t+s-1|-1))\Big),$$

observing $|t + s - 1| - 1 \leq |s - t| + |2t - 1| - 1$ such that the second exponential asymptotically only contributes at the corners $(0,0)$ and $(1,1)$ of the diagonal.

We shall see, however, that for the Hilbert-Schmidt norm evaluation we face $(\delta_n^\sigma)^2$ as the operator kernel, which also behaves like a smoothing kernel on the diagonal, but needs to be rescaled by $\|\delta_n^\sigma\|_{L^2}^2 = (1 + \mathcal{O}(1))\sqrt{n}/(4\sigma^3)$.

Consequently, in terms of $\Delta_n = \mathrm{diag}(\Delta_n^{\bar{\Lambda}_{ii}})_{i=1,\dots,d}$ and its kernel $\delta_n(t,s) =$

$\mathrm{diag}(\delta_n^{\bar{\Lambda}_{ii}}(t,s))_{i=1,\ldots,d}$, the Fisher information evaluates as

$$
\begin{aligned}
I_n^Q &= \frac{1}{2}\|Q_{n,0}^{-1/2}\dot{Q}_0 Q_{n,0}^{-1/2}\|_{HS}^2 \\
&= \frac{1}{2}\,\mathrm{trace}(T_r I Q_{n,0}^{-1} I^* T_r^* \mathcal{M}_M T_r I Q_{n,0}^{-1} I^* T_r^* \mathcal{M}_M) \\
&= \frac{1}{2}\,\mathrm{trace}((T_r \Delta_n T_r^*)\mathcal{M}_M (T_r \Delta_n T_r^*)\mathcal{M}_M) \\
&= \frac{1}{2}\int_0^1 \int_0^1 \mathrm{trace}_{\mathbb{R}^{d\times d}}(\delta_n(r(t),r(s))M(s)\delta_n(r(s),r(t))M(t))\,dtds.
\end{aligned}
$$

We now use $\int_0^1 M(s)a_n e^{-|t-s|a_n}ds = 2M(t)(1+\mathcal{o}(1))$ uniformly over $t \in [b_n, 1-b_n]$ whenever $a_n \to \infty$, $a_n b_n \to \infty$ and $M(t)$ is continuously differentiable. Together with the asymptotic behaviour of δ_n^σ we obtain

$$
\begin{aligned}
&\int_0^1 \delta_n^{\bar{\Lambda}_{ii}}(r_i(t),r_i(s))M_{ij}(s)\delta_n^{\bar{\Lambda}_{jj}}(r_j(s),r_j(t))\,ds \\
&= (1+\mathcal{o}(1))\frac{n}{4\bar{\Lambda}_{ii}\bar{\Lambda}_{jj}}\int_0^1 \exp(-\sqrt{n}(\bar{\Lambda}_{ii}r_i'(t)+\bar{\Lambda}_{jj}r_j'(t))|t-s|)M_{ij}(s)\,ds \\
&= (1+\mathcal{o}(1))\frac{\sqrt{n}}{2\bar{\Lambda}_{ii}\bar{\Lambda}_{jj}(\bar{\Lambda}_{ii}r_i'(t)+\bar{\Lambda}_{jj}r_j'(t))}M_{ij}(t) \\
&= \frac{\sqrt{n}M_{ij}(t)(1+\mathcal{o}(1))}{2\bar{\Lambda}_{ii}\bar{\Lambda}_{jj}(\lambda_i(t)+\lambda_j(t))}
\end{aligned}
$$

with $\mathcal{o}(1)$ uniformly in n and $t \in [n^{-p}, 1-n^{-p}]$ for any $p \in (0,1/2)$ to infer

$$
\begin{aligned}
I_n^Q &= \frac{\sqrt{n}}{4}\sum_{i,j=1}^d \bar{\Lambda}_{ii}^{-1}\bar{\Lambda}_{jj}^{-1}\int_0^1 (\lambda_i(t)+\lambda_j(t))^{-1}M_{ij}(t)^2(1+\mathcal{o}(1))dt \\
&= \frac{\sqrt{n}(1+\mathcal{o}(1))}{4}\int_0^1 \sum_{i,j=1}^d \frac{(O\mathbb{H}O^\top)_{ij}^2}{\lambda_i(\lambda_i+\lambda_j)\lambda_j}(t)\,dt.
\end{aligned}
$$

Asymptotically for $n \to \infty$ neglecting terms of smaller order, this bound is obtained by the worst parametric perturbation $\mathbb{H}^*(t) = \Sigma_0 A \Sigma_0^{1/2} + \Sigma_0^{1/2}A\Sigma_0$, which we evaluate using duality with respect to the scalar product $\int_0^1 \sum_{i,j} A_{ij}(t)B_{ij}(t)dt$ as

$$
\sup_{\mathbb{H}:\mathbb{H}(t)=\mathbb{H}(t)^\top} \frac{(\int_0^1 \sum_{i,j=1}^d A_{ij}(t)\mathbb{H}_{ij}(t)\,dt)^2}{\frac{\sqrt{n}}{4}\int_0^1 \sum_{k,l=1}^d \frac{(O\mathbb{H}O^\top)_{kl}^2}{\lambda_k(\lambda_k+\lambda_l)\lambda_l}(t)\,dt} = \frac{(\int_0^1 \sum_{i,j=1}^d A_{ij}(t)\mathbb{H}_{ij}^*(t)\,dt)^2}{\frac{\sqrt{n}}{4}\int_0^1 \sum_{k,l=1}^d \frac{(O\mathbb{H}^*O^\top)_{kl}^2}{\lambda_k(\lambda_k+\lambda_l)\lambda_l}(t)\,dt}
$$

$$
= \frac{4}{\sqrt{n}}\int_0^1 \langle A(t),(\Sigma A\Sigma^{1/2}+\Sigma^{1/2}A\Sigma)(t)\rangle_{HS}\,dt.
$$

Finally, remark that the Cramér-Rao inequality, e.g. [18, Thm. 2.5.10], is applicable since $(\mathbf{N}(0, Q_{n,\varepsilon}))_\varepsilon$ forms an exponential family in $(Q_{n,\varepsilon}^{-1})_\varepsilon$, which is differentiable at $\varepsilon = 0$, and thus the models $(\mathbf{N}(0, Q_{n,\varepsilon}))_\varepsilon$ as well as $(\mathbf{N}(0, C_{n,\varepsilon}))_\varepsilon$ are regular.

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Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin, Germany
E-mail: bibinger@math.hu-berlin.de

School of Business and Economics
Humboldt-Universität zu Berlin
Spandauer Str. 1
10178 Berlin, Germany
E-mail: malecpet@hu-berlin.de

School of Business and Economics
Humboldt-Universität zu Berlin
Spandauer Str. 1
10178 Berlin, Germany
E-mail: nikolaus.hautsch@wiwi.hu-berlin.de

Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6
10099 Berlin, Germany
E-mail: mreiss@math.hu-berlin.de